HOMOLOGICAL PROPERTIES OF REPRESENTATIONS OF P-ADIC GROUPS RELATED TO GEOMETRY OF THE GROUP AT INFINITY

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Abstract.

Let G be a reductive group with compact center over a nonarchimedian local field, and let X be its Bruhat-Tits building.

The central result of the first part of the work establishes a connection between the direct image of an equivariant sheaf on X to the Borel-Serre compactification with Jacquet functor. More precisely, we prove that there exists an exact functor L on a category of representations of the group, such that the following holds. Let F be an equivariant simplicial sheaf on X; then the restriction of the direct image of F to the boundary of the Borel-Serre compactification is canonically isomorphic to $L(\Gamma(F))$ (here Γ stands for global sections). Furthermore, the restriction of L to the category of admissible representations is described explicitly in terms of Jacquet functor.

As an application we obtain a canonical isomorphism between the homological duality on the derived category of smooth G-modules, and the composition of the Deligne-Lusztig functor with the Grothendieck-Serre duality (the latter comes from the action of the Bernstein center on every smooth G-module). (This isomorphism was formulated by J. Bernstein; another proof was recently obtained by P. Schneider).

In the second part I prove a conjecture by D. Kazhdan which expresses the Euler characteristic of Yoneda Ext's between two admissible representations as the integral of the product of their characters over the set of elliptic conjugacy classes (another proof of this result was recently obtained by P. Schneider).

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Introduction

1. Let F be a non-archimedian local field, and $G = \underline{G}(F)$ be the group of F-points of a reductive algebraic group $\underline{G}$ defined over F. We assume for convenience that G has compact center.

We fix an algebraically closed coefficient field k of characteristic 0. $\mathcal{H}$ will denote the Hecke algebra of k-valued locally constant compactly supported measures on G.

The main object of our study is the category $\mathcal{M}$ of finitely generated smooth representations of G over k [BZ].

2. In chapter 1 the category $\mathcal{M}$ is studied using equivariant sheaves on the Bruhat-Tits building X of G.

For a G-space Y we will denote by $Sh_G(Y)$ the category of G-equivariant sheaves on X. Let $\mathcal{S}\mathfrak{h} \subset Sh_G(X)$ be the category of simplicial G-equivariant sheaves of finite-dimensional vector-spaces on X. We have the derived functor of sections with compact support $R\Gamma_c : D^b(\mathcal{S}\mathfrak{h}) \to D^b(\mathcal{M})$, and Verdier dual functor $R\Gamma : D^b(\mathcal{S}\mathfrak{h}) \to D^b(\mathcal{M}^{opp})$ (where $\mathcal{M}^{opp}$ is the opposite category to $\mathcal{M}$).

Consider the imbedding $j : X \to \overline{X}$ of X into its Borel-Serre compactification $\overline{X}$ [BoSer], and the equivariant direct image functor $j_*^G : D^b(\mathcal{S}\mathfrak{h}) \to D^b(Sh_G(\overline{X}))$ (discussed below). Let i stand for the imbedding $i : \overline{X} - X \hookrightarrow \overline{X}$. The main technical result of chapter 1 is existence of a functor $\mathcal{L} : \mathcal{M}^{opp} \to Sh_G(\overline{X} - X)$ such that:

i) We have $i^* \circ j_*^G = \mathcal{L} \circ \Gamma$ canonically.

ii) $\mathcal{L}$ is exact.

iii) Let $\mathcal{R} \subset \mathcal{M}$ be the subcategory of admissible representations. Then $\mathcal{L}|_{\mathcal{R}^{opp} \cong \mathcal{R}}$ is a well-known combinatorially defined functor: for $M \in \mathcal{R}$ and $y \in \overline{X} - X$ the stalk of $\mathcal{L}(M)$ at y is the Jacquet functor $r_{P_y}(M)$ where $P_y = Stab_G(y)$ is a parabolic subgroup in G. (The equivalence between $\mathcal{R}$ and $\mathcal{R}^{opp}$ used above sends a representation $\rho \in \mathcal{R}$ to its contragradient $\rho\check{}$).

We remark that construction of $\mathcal{L}$ and proof of i) and iii) are rather straightforward, while the proof of ii) required some effort.

To apply this result to representation theory one needs to be able to "localize" representations to sheaves on the building. An ideal result in this direction would be a one-sided inverse to the functor $R\Gamma_c$. Unfortunately such functor is unavailable at present. By the results of [Sch-St1], [Sch-St2] the full subcategory $D^b(\mathcal{R}) \subset D^b(\mathcal{M})$ lies in the image of $R\Gamma_c$. (And also $D^b(\mathcal{M})$ lies in the image of the functor $R\Gamma^c$ from G-equivariant simplicial sheaves of possibly infinite dimensional vector spaces to $D^b(\widetilde{\mathcal{M}})$, where $\widetilde{\mathcal{M}}$ is the larger category of *all* (not necessary finitely generated) smooth G-modules).

The result of [Sch-St2] is sufficient for our application. By a "general nonsense" argument we get another localization type claim sufficient for us (but much less explicit, and thus probably less useful than the Schneider-Stuhler's result). It says the following: for any object $B \in D^b(\mathcal{M})$ and $i \in \mathbb{Z}$ one can find an object $A \in D^b(\mathcal{S}\mathfrak{h})$ and a morphism $R\Gamma_c(A) \to B$ which induces isomorphism on H^j for $j > i$.

Using the connection between the Verdier duality on sheaves with the homological duality on representations (also observed in Schneider's work [Schn2]) one

derives the following statement (which belongs essentially to J. Bernstein and was the starting point for this part of the work): there exists a canonical isomorphism of functors:

$$D_h \cong DL \circ D_{Gr}$$

where DL is the Deligne-Lusztig functor [DL]; D_h is the homological duality $D_h : M \to RHom_G(M, C_c^\infty(G))$; and $D_{Gr}(M)$ is the Grothendieck dual of M as a module over the Bernstein center [BD] (for admissible M we have $D_{Gr}(M) = M\check{}$ where $M\check{}$ is the contragradient module). (See the main text for detailed definitions).

A proof of the latter isomorphism using (co)sheaves on the Borel-Serre compactification of the Bruhat-Tits building appears also in [Sch-St2].

3. The starting point for Chapter 2 was an attempt to find an "a'priori" proof for the following conjecture by Kazhdan (proved recently in [Sch-St2] by making use of an explicit resolution of a G-module constructed in [Sch-St2], [Sch-St1]).

Let Ell be the set of regular semisimple elliptic conjugacy classes in G. Then Ell carries a canonical measure (an analogue of the Weyl integration measure for compact Lie groups, see section 4 of chapter 2 for a recollection), which we denote by $d\mu$. For an admissible G-module ρ let χ_ρ denote its character, and let us use the same notation for the corresponding locally constant function on Ell. By [K] we have $\chi_\rho \in L^2(Ell, d\mu)$ provided the base local field F has characteristic 0. Let now ρ_1, ρ_2 be two admissible representations. Then the conjecture (= Theorem 0.20 of chapter 2) says that

$$\sum (-1)^i \dim Ext^i(\rho_1, \rho_2) = \int_{Ell} \chi_{\rho_1}(g^{-1})\chi_{\rho_2}(g)d\mu(g) \tag{1}$$

Recall the following construction (see e.g. [Br], §IX.2 or [Vig] §2 and references therein). Let $\mathcal{A}$ be a Noetherian associative algebra with 1, having finite homological dimension. Then to a pair (M, E) where M is a finitely generated $\mathcal{A}$-module, and $E \in End(M)$ one can assosiate an element of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$, denoted by $Tr_{H-St}(M, E)$, and called the *Hattori-Stallings trace* of (M, E). This map enjoys the properties:

a) $Tr_{H-St}(M, E_1 \circ E_2) = Tr_{H-St}(N, E_2 \circ E_1)$ for two modules M, N as above, and $E_1 \in Hom(N, M)$, $E_2 \in Hom(M, N)$;

b) $Tr(\mathcal{A}, R(a)) = a \mod [\mathcal{A}, \mathcal{A}]$ for $a \in \mathcal{A}$ where $R(a)$ is the right multiplication by $a \in A$ considered as an endomorphism of the free module;

c) additivity on short exact sequences.

It is characterized by these properties uniquely.

Applying this construction (with slight modifications necessary because $\mathcal{H}$ is not unital, see 0.21 of chapter 2) to our situation we obtain a map $Tr_{H-St} : \{(\mathfrak{M}, E) \mid \mathfrak{M} \in \mathcal{M}, E \in End(\mathfrak{M})\} \to \mathcal{H}/[\mathcal{H}, \mathcal{H}]$ satisfying above properties a), b), c). For $\mathfrak{M} \in \mathcal{M}$ the element $Tr_{H-St}(\mathfrak{M}, Id)$ is called a pseudo-coefficient or rank function of $\mathfrak{M}$; we will denote it by $\langle \mathfrak{M} \rangle$.

Recall that for $h \in \mathcal{H}$ and a regular elliptic element $g \in G$ the *orbital integral* $O_g(h) \in k$ is defined by

$$O_x(h) := \int\limits_{y \in G} \frac{h}{\nu}(yxy^{-1})d\nu(y),$$

where ν is a Haar measure on G; the resulting expression does not depend on the choice of ν.

A formal argument shows (see section 4 of chapter 2) that (1) is equivalent to the equality

$$O_{g^{-1}}(\langle\rho\rangle) = \chi_\rho(g) \tag{2}$$

being true for any admissible representation ρ and regular elliptic element $g \in G$.

Notice that the latter statement makes sense (and will be proved) without the restriction $\mathrm{char}(F) = 0$ or $k = \mathbb{C}$.

If ρ is cuspidal, and $m_\rho \in \mathcal{H}$ is its matrix coefficient normalized by $\int m_\rho \cdot \chi_\rho = 1$, then $\langle\rho\rangle = m_\rho \mod [\mathcal{H}, \mathcal{H}]$. Thus (2) generalizes a well-known equality going back to Harish-Chandra

$$\chi_\rho(g) = O_{g^{-1}}(m_\rho).$$

To prove (2) we extend the functional $Tr(g, \rho) = \chi_\rho(g)$ from admissible to arbitrary smooth representations ρ, and then show (Theorem 3.1) that for a projective $\rho \cong \mathcal{H}^{\oplus n}E$, where $E \in Mat_n(\mathcal{H})$ is an idempotent we have

$$Tr(g, \rho) = O_{g^{-1}}(\sum E_{ii}). \tag{3}$$

The intuitive idea behind the definition of $Tr(g, \mathfrak{M})$ is that for a small open compact subgroup $K \subset G$ one can find a finite dimensional g-invariant subspace $M_0 \subset \mathfrak{M}^K$ such that the quotient $\mathfrak{M}^K/M_0$ "comes from the parabolic induction"; thus it is reasonable to set $Tr(g, \mathfrak{M}) = tr(g, M_0)$. Realization of this idea exploits certain natural (multi)filtration on the Hecke algebra connected with the action of $\mathcal{H}$ on the principal series (see section 2 of chapter 2).

To prove (3) we relate our filtration to the one defined "geometrically" in terms of support of a distribution $h \in \mathcal{H}$. Here some geometry of G and its asymptotic cones $(G/U \times G/U^-)/L$ at infinity is used.

1. Sheaves on the building and representations

Preliminaries and notations. In both chapters we will abuse the terminology by saying "a parabolic/Levi subgroup" instead of "the group of F-points of an F-rational parabolic subgroup/Levi subgroup of a proper parabolic subgroup". By rank of a reductive group defined over F we mean its split rank.

0.1. k is a fixed coefficient field of characteristic 0; for a set S we denote by $k[S]$ the space $\bigoplus_S k$. For a topological space Z we write $C^\infty(Z)$ and $C_c^\infty(Z)$ for the space of locally constant functions on Z and locally constant functions with compact support on Z. If Z is an open subset of a group, we denote by $\mathcal{H}(Z)$ the space of locally constant measures on Z with compact support; if Z is biinvariant under the action of a compact open subgroup $K \subset G$, then $\mathcal{H}(Z,K) \subset \mathcal{H}(Z)$ is the subspace of K-biinvariant measures. If Z is a semigroup, then $\mathcal{H}(Z)$ and $\mathcal{H}(Z,K)$ carry the structure of an associative algebra provided by convolution.

By $\underline{W}$ or $\underline{W}_Z$ we will denote the constant sheaf on Z with stalk W; if $\iota : Z \to X$ is a closed embedding then we will also write $\underline{W}_Z$ instead of $\iota_*(\underline{W}_Z)$. By $\underline{Hom}$ we denote internal Hom in the category of (G-equivariant) sheaves.

For a G-space X let $Sh_G(X)$ denote the category of G-equivariant sheaves on X. Recall that for arbitrary topological group G and a G-space X a G-equivariant sheaf is the same as a sheaf $\mathcal{F}$ on X equipped with a G-action such that each section of $\mathcal{F}$ is locally constant. Here a section $s \in \Gamma(U,\mathcal{F})$ is called locally constant if for $x \in U$ there exist a neigborhood U' of x and a neigborhood V of identity $e \in G$ such that $g(s)|_{U' \cap g(U')} = s|_{U' \cap g(U')}$ for $g \in V$.

For a morphism of G-spaces $p : X \to Y$ we have the functor of inverse image $p^* : Sh_G(Y) \to Sh_G(X)$ and the adjoint functor $p_*^G : Sh_G(X) \to Sh_G(Y)$. Note that the inverse image commutes with the forgetful functor from equivariant sheaves to sheaves while the direct image does not: for $\mathcal{F} \in Sh_G(X)$ one can describe $p_*^G(\mathcal{F})$ as the sheaf of locally constant sections of $p_*(\mathcal{F})$ (see [Schn1] for a detailed discussion).

From now on let X denote the (semi-simple) building of G, unless stated otherwise. Let $\overline{X}$ be the Borel-Serre compactification of X [BorSe]; put $Y = \overline{X} - X$, and denote the imbeddings by $j : X \hookrightarrow \overline{X}$ and $i : Y \hookrightarrow \overline{X}$. Recall that Y is the spherical building of G. For a parabolic $P \subset G$ we let $\Delta_P \subset Y$ denote the corresponding simplex.

We will often say polysimplex meaning a polysimplex of the canonical polytriangulation of X (facette in the terminology of [BT1]).

Let $\mathcal{S}\mathfrak{h}$ be the full subcategory of $Sh_G(X)$ consisting of sheaves of finite dimensional vector spaces which are constant on every (poly)simplex.

0.2. By a representation we will mean a representation in a vector space over a field k; the field will be endowed with discrete topology.

Let $\widetilde{\mathcal{M}}$ be the category of smooth G-modules; let $\widetilde{\mathcal{M}}\check{}$ be the dual category. It will be convenient to use an explicit realization of $\widetilde{\mathcal{M}}\check{}$. Namely $\widetilde{\mathcal{M}}\check{}$ is identified with the category of complete topological vector spaces V having a basis of neighbourhoods of 0 consisting of vector subspaces of finite codimension, together with a continuous homomorphism $\pi : G \to \mathrm{Aut}(V)$. (The group $\mathrm{Aut}(V)$ is endowed with the topology of pointwise convergence.) The antiequivalence $* : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}\check{}$ sends a G-module M to its full linear dual equipped with the pointwise-convergence topology (=

the inverse limit topology on $M^* = \varprojlim V^*$, where V runs over the set of finite dimensional subspaces in M). The inverse antiequivalence (also denoted by $*$) sends a topological G-module L to the space of continuous linear functionals $L \to k$.

Let $\mathcal{M} \subset \widetilde{\mathcal{M}}$ (respectively $\mathcal{R} \subset \mathcal{M}$) be the full subcategory of finitely generated smooth (respectively admissible) G-modules. Let $\mathcal{M}\check{} \subset \widetilde{\mathcal{M}}\check{}$ and $\mathcal{R}\check{} \subset \mathcal{M}\check{}$ be the dual categories.

We have an exact functor $Sm : \widetilde{\mathcal{M}}\check{} \to \widetilde{\mathcal{M}}$ sending a module $M \in \widetilde{\mathcal{M}}\check{}$ to the module of smooth vectors in M. The functor Sm induces an equivalence $\mathcal{R}\check{} \widetilde{\to} \mathcal{R}$. (Of course it does *not* send the category $\mathcal{M}\check{}$ to $\mathcal{M}$.) For $M \in \mathcal{M}$ we let $\mathcal{M}\check{} = Sm(*(M)) \in \widetilde{\mathcal{M}}$ be its contragradient.

We have left-exact functors $\Gamma_c : Sh \to \mathcal{M}$ and $\Gamma : Sh \to \mathcal{M}\check{}$, where Γ stands for global sections, and Γ_c stands for sections with compact support. The linear topology on the space $\Gamma(X, \mathcal{F})$ is the inverse limit topology on $\Gamma(X, \mathcal{F}) = \varprojlim_K \Gamma(K, \mathcal{F})$ where K runs over all compact subsets of X.

As we will soon recall, $\mathcal{S}\mathfrak{h}$ has enough injectives (and projectives). Hence the right derived functors $R\Gamma : D^*(\mathcal{S}\mathfrak{h}) \to D^*(\widetilde{\mathcal{M}})$ and $R\Gamma_c : D^*(\mathcal{S}\mathfrak{h}) \to D^*(\mathcal{M})$ where $* = b, +$ are defined.

1. **Verdier duality for equivariant sheaves and homological duality.** Here we recall some basic generalities on homological algebra of $\mathcal{S}\mathfrak{h}$, including Verdier duality and its connection with homological duality on representations. This section does not contain new results: Proposition 1.3a) follows from [Schn1], §3 and remark before [Schn2], Proposition 2; our 1.3b) is Proposition 4 in [Schn2]. We sketch the proof for completeness.

1.1. Let $Sh(X)$ be the category of sheaves of finite dimensional k-vector spaces on X constant on any polysimplex of the canonical polytriangulation. Then $Sh(X)$ has enough injectives; the injective generators are the constant sheaves $\underline{k}_{\overline{\Delta}}$ on the closure of a simplex. From the fact that the stabilizer of any point of X is compact, and hence its smooth representations form a semisimple category, it is easy to deduce that a sheaf $\mathcal{F} \in \mathcal{S}\mathfrak{h}$ is injective iff it is injective as an object of $Sh(X)$. For a polysimplex $\Delta \subset X$ and a smooth finite dimensional representation R of the stabilizer of Δ let $\mathcal{F}_{\Delta,R} \in \mathcal{S}\mathfrak{h}$ be the G-equivariant sheaf $\mathcal{F}_{\Delta,R} := \prod_{g \in G/Stab(\Delta)} \underline{k}_{g\overline{\Delta}} \otimes C^\infty(g \cdot Stab(\Delta)) \bigotimes_{Stab(\Delta)} R$. Obviously $\mathcal{F}_{\Delta,R}$ is injective.

Let $\mathfrak{I}$ be the category of injective objects in $Sh(X)$, and $\mathfrak{I}_G$ be the category of injective objects in $\mathcal{S}\mathfrak{h}$.

Let us call a module $M \in \mathcal{M}\check{}$ (respectively $M \in \mathcal{M}$) standard injective (respectively projective) if it is a direct sum of modules induced (respectively compactly induced) from smooth finite dimensional representations of open compact subgroups.

Lemma 1.2. *a) $\mathcal{S}\mathfrak{h}$ is of finite homological dimension and has enough injectives. Any injective object in $\mathcal{S}\mathfrak{h}$ is a direct sum of sheaves of the form $\mathcal{F}_{\Delta,R}$.*

b) $\Gamma : \mathcal{S}\mathfrak{h} \to \mathcal{M}\check{}$ sends injectives to injectives and $\Gamma_c : \mathcal{S}\mathfrak{h} \to \mathcal{M}$ sends injectives to projectives.

c) An injective (respectively projective) object of $\mathcal{M}\check{}$ (respectively $\mathcal{M}$) is isomorphic to $\Gamma(\mathcal{F})$ (respectively $\Gamma_c(\mathcal{F})$) for $\mathcal{F} \in \mathfrak{I}_G$ iff it is a standard injective (projective).

d) If $\mathcal{F},\mathcal{G}\in\mathfrak{I}_G$ then $\underline{Hom}(\mathcal{F},\mathcal{G})$ and $\mathcal{F}\otimes\mathcal{G}$ are also injective.

Proof Clear. □

We see that any left-exact functor on $\mathcal{S}\mathfrak{h}$ admits the (right) derived functor.

Proposition 1.3. *a) There exists an object $\underline{V}\in D^b(\mathcal{S}\mathfrak{h})$ equipped with an isomorphism of functors on $D^b(\mathcal{S}\mathfrak{h})$*

$$R\Gamma\circ R\underline{Hom}(\_,\underline{V})\cong *\circ R\Gamma_c \tag{4}$$

Isomorphism (4) *defines $\underline{V}$ uniquely up to the unique isomorphism. $\underline{V}$ is called the equivariant dualizing sheaf, and the functor $\mathbb{V}:=R\underline{Hom}(\_,\underline{V})$ is called the equivariant Verdier duality.*

b) Let $D_h: D^b(\mathcal{M})\to D^b(\mathcal{M})$ be the homological duality defined by $D_h(M)=RHom(M,\mathcal{H})$. We have a canonical isomorphism:

$$R\Gamma_c\circ\mathbb{V}\cong D_h\circ R\Gamma_c \tag{5}$$

Proof a) Obviously $Hom_{\mathcal{S}\mathfrak{h}}(\mathcal{F},\mathcal{G})=(\Gamma(\underline{Hom}(\mathcal{F},\mathcal{G})))^G$. Using Lemma 1.2b),d) we see that $RHom(\mathcal{F},\mathcal{G})=RI\circ R\Gamma(R\underline{Hom}(\mathcal{F},\mathcal{G}))$ where I stands for invariants $I:\mathcal{M}\check{}\to Vect$. So (4) yields an isomorphism of functors $Hom(\_,\underline{V})\cong H^0(RI\circ *\circ R\Gamma_c)$; uniqueness of $\underline{V}$ follows.

To show existence we adapt a standard construction of the dualizing sheaf to our situation. For a polysimplex $\iota:\Delta\hookrightarrow X$ let C_Δ be the standard resolution of $\iota_!(\underline{k}[\dim(\Delta)])$ (so we have $C^i_\Delta=\bigoplus\limits_{\Delta'\subset\overline{\Delta};\dim(\Delta')=-i}\underline{k}_{\overline{\Delta'}}$).

We set $\mathbb{V}(\underline{k}_{\overline{\Delta}})=C_\Delta$; this obviously extends to a contravariant functor $Kom(\mathfrak{I})\to Kom(\mathfrak{I})$ and defines a contravariant functor $Kom(\mathfrak{I}_G)\to Kom(\mathfrak{I}_G)$. (We have $\mathbb{V}(\mathcal{F}_{\Delta,R})=\prod\limits_{g\in G/Stab(\Delta)}C_{g\Delta}\otimes C^\infty(g\cdot Stab(\Delta))\bigotimes\limits_{Stab(\Delta)}R^*$.) The induced functor $D^b(\mathcal{S}\mathfrak{h})\to D^b(\mathcal{S}\mathfrak{h})$ is also denoted by $\mathbb{V}$. One can easily define isomorphism of complexes $\mathbb{V}(\mathcal{F}\otimes\mathcal{G})\cong\underline{Hom}(\mathcal{F},\mathbb{V}(\mathcal{G}))$ for $\mathcal{F},\mathcal{G}\in\mathfrak{I}_G$; the isomorphism of bifunctors on $D^b(\mathcal{S}\mathfrak{h})$ follows. Hence $\mathbb{V}$ is corepresented by $\underline{V}=\mathbb{V}(\underline{k})$.

For $\mathcal{F}\in\mathfrak{I}_G$ we have a functorial map of complexes $\Gamma(\mathbb{V}(\mathcal{F}))\to *(\Gamma_c(\mathcal{F}))$ which is a quasiisomorphism; here the pairing $\langle\ ,\ \rangle:\Gamma_c(\mathcal{F})\times\Gamma(\mathbb{V}(\mathcal{F}))\to k$ is the standard Verdier pairing. This yields isomorphism (4).

b) The pairing $\langle\ ,\ \rangle$ gives a map $\phi:\Gamma(\mathbb{V}(\mathcal{F}))\to Hom_G(\Gamma_c(\mathcal{F}),C^\infty(G))$; here $(\phi(y)(x))(g)=\langle x,g(y)\rangle$. It is clear that $\phi(\Gamma_c(\mathbb{V}(\mathcal{F})))\subset Hom_G(\Gamma_c(\mathcal{F}),C_c^\infty(G))\subset Hom_G(\Gamma_c(\mathcal{F}),C^\infty(G))=\Gamma(\mathbb{V}(\mathcal{F}))$; more precisely for $\mathcal{F}\in\mathfrak{I}_G$ we get a functorial quasiisomorphism $\Gamma_c(\mathbb{V}(\mathcal{F})))\to Hom_G(\Gamma_c(\mathcal{F}),C_c^\infty(G))$ which yields (5). (In fact for $\mathcal{F}\in\mathfrak{I}_G$ isomorphisms (4), (5) boil down to well-known equalities

$$*(ind_K^G(R))=Ind_K^G(R^*),$$

$$Hom_G(ind_K^G(R),C_c^\infty(G))=ind_K^G(R^*).)$$

Proposition is proved. □

Remark 1.4. In [Schn2] it is proved that $\underline{V}$ is actually concentrated in one homological degree; we will not use this fact here.

2. **A "localization" type theorem.** Let us call a morphism in $D^b(Sh)$ *h-quasiisomorphism* (respectively *hc-quasiisomorphism*) if it induces an isomorphism on $R\Gamma$ (respectively $R\Gamma_c$).

Let $D^0(\mathcal{M}\check{\ })$, $D^0(\mathcal{M})$ be the full subcategories in $D^b(\mathcal{M}\check{\ })$, $D^b(\mathcal{M})$ respectively consisting of objects quasiisomorphic to a finite complex of standard injectives (respectively projectives). $LD^b(\mathcal{S}\mathfrak{h})$, $L^0D^b(\mathcal{S}\mathfrak{h})$ be the localizations of $D^b(\mathcal{S}\mathfrak{h})$ respectively with respect to h-quasiisomorphism and hc-quasiisomorphisms.

Proposition 2.1. *$R\Gamma$ provides an equivalence between $LD^b(\mathcal{S}\mathfrak{h})$ and $D^0(\mathcal{M}\check{\ })$.*
$R\Gamma_c$ provides an equivalence between $L^0D^b(\mathcal{S}\mathfrak{h})$ and $D^0(\mathcal{M})$.

For the proof of the Proposition we need the following

Lemma 2.2. *Assume that $\mathcal{F}, \mathcal{G} \in \mathcal{S}\mathfrak{h}$ and $\mathcal{F}$ is injective; let a morphism $\phi : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{G})$ be given. Then there exist a sheaf $\mathcal{F}' \in \mathcal{S}\mathfrak{h}$ and morphisms $s : \mathcal{F}' \to \mathcal{F}$, $\phi' : \mathcal{F}' \to \mathcal{G}$ such that s is an h-quasiisomorphism, and $\Gamma(\phi') = \phi \circ \Gamma(s)$.*

Proof: First notice that $X \times X$ is a union of poly-simplicial subspaces X_r such that $pr_i|_{X_r}$ is proper for $i = 1, 2$ and $pr_1|_{X_r}$ has geodesically contractible fibers. (To see this let us choose an apartment $\mathfrak{A} \subset X$, an open simplex $\Delta \subset \mathfrak{A}$, and exaust $\mathfrak{A}$ by simplicial convex neigborhoods $\mathfrak{C}_r$ of $\overline{\Delta}$. Then take $X_r = G(\overline{\Delta} \times \mathfrak{C}_r)$. The desired properties of X_r are clear. We have $\bigcup\limits_r X_r = X \times X$ because $G(\overline{\Delta}) = X$ and any two points of X are G-conjugate to two points lying in $\mathfrak{A}$.) Let $p_i = pr_i|_{X_r}$, $i = 1, 2$. Set $\mathcal{F}_r = p_{2*}p_1^*\mathcal{F}$.

We claim that for large enough r we can take $\mathcal{F}' = \mathcal{F}_r$.

It is clear that if $\mathcal{F}$ is injective then $R^ip_{2*}p_1^*\mathcal{F} = 0$ for $i \neq 0$. Hence $R\Gamma(\mathcal{F}_r) = R\Gamma(p_1^*\mathcal{F}) = R\Gamma(\mathcal{F})$ since p_1 has contractible fibers. Applying p_{2*} to the morphism $p_1^*\mathcal{F} \to \delta_*(\mathcal{F})$ where $\delta : X \to X \times X$ is the diagonal imbedding we get a map $\mathcal{F}_r \to \mathcal{F}$ which induces isomorphism on cohomology.

It remains to construct $\phi' : \mathcal{F}_r \to \mathcal{G}$ for large r.

For any $x \in X$ consider the composition $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{G}) \to \mathcal{G}_x$ where $\mathcal{G}_x$ is the stalk of $\mathcal{G}$ at x. It is continuous and $\mathcal{G}_x$ is discrete; hence it factors through a map $\Gamma(\mathcal{F}) \to \Gamma(\mathfrak{C}, \mathcal{F}) \to \mathcal{G}_x$ for some compact $\mathfrak{C} = \mathfrak{C}(x) \subset X$. Then clearly the same is true for any y lying in one polysimplex with x with the same compact set $\mathfrak{C}(y) = \mathfrak{C}(x)$.

We can take a finite number of polysimplicies Δ_i such that $G(\cup\Delta_i) = X$; and then find r such that $X_r \supset \mathfrak{C}(x_i) \times \Delta_i$ for $x_i \in \Delta_i$. Then we see that the map $\Gamma(\mathcal{F}_r) = \Gamma(\mathcal{F}) \to \mathcal{G}_x$ factors through $\Gamma(\mathcal{F}_r) \to (\mathcal{F}_r)_x$ for all $x \in X$, i.e. the composition $\underline{k} \otimes \Gamma(\mathcal{F}_r) \to \underline{k} \otimes \Gamma(\mathcal{G}) \to \mathcal{G}$ factors through the canonical surjection $\underline{k} \otimes \Gamma(F_r) \to \mathcal{F}_r$. The Lemma is proved. □

Proof of the Proposition. First note that image of $R\Gamma$ is contained in $D^0(\mathcal{M}\check{\ })$ since an injective sheaf goes to a standard injective module.

Lemma 2.2 implies that for $\mathcal{F}, \mathcal{G} \in \mathfrak{I}_G$ the map $\mathrm{Hom}_{LD^b(\mathcal{S}\mathfrak{h})}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{M}\check{\ }}(R\Gamma\mathcal{F}, R\Gamma\mathcal{G}) = \mathrm{Hom}_{\mathcal{M}\check{\ }}(\Gamma\mathcal{F}, \Gamma\mathcal{G})$ is surjective.

It is also injective for the following reason. Consider the commutative square

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "s"] & \mathcal{G} \\
\underline{\Gamma(\mathcal{F})} \arrow[u] \arrow[r, "\underline{\Gamma(s)}"] & \underline{\Gamma(\mathcal{G})} \arrow[u]
\end{tikzcd}$$

where $\Gamma(s) : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{G})$ is the map on global sections induced by s, and $\underline{\Gamma(s)}$ is the corresponding map of constant sheaves. The left vertical arrow in this diagram is surjective for injective $\mathcal{F}$, so we have $\Gamma(s) = 0$, $\mathcal{F} \in \mathfrak{I}_G \Rightarrow s = 0$.

Thus $R\Gamma$ induces an isomorphism $\mathrm{Hom}_{LD^b(\mathcal{S}\mathfrak{h})}(\mathcal{F},\mathcal{G}) \widetilde{\longrightarrow} \mathrm{Hom}_{\mathcal{M}\check{}}(R\Gamma\mathcal{F}, R\Gamma\mathcal{G})$ for $\mathcal{F},\mathcal{G}\in\mathfrak{I}_G$, and hence for any $\mathcal{F}$, $\mathcal{G}\in D^b(\mathcal{S}\mathfrak{h})$.

It remains to check that $R\Gamma : LD^b(\mathcal{S}\mathfrak{h}) \to D^0(\mathcal{M}\check{})$ is surjective on isomorphism classes of objects. Let C be a finite complex of standard injectives $0\to I_a \to ... \to I_b \to 0$. We then have some injective sheaves $\mathcal{F}_i \in \mathcal{S}\mathfrak{h}$ with $\Gamma(\mathcal{F}_i)\cong I_i$. We now use inductively Lemma 2.2 to construct a complex of sheaves $\tilde{C} = 0\to \mathcal{F}'_a \to .. \to \mathcal{F}'_b \to 0$, where $\mathcal{F}'_i = (\mathcal{F}_i)_{r_i}$, $H^j(F'_i)=0$ for $j\neq 0$ and $\Gamma(\tilde{C})\cong C$.

The first statement is proved, and the second one follows by Verdier duality. □

Remark 2.3. I do not know whether in general $D^0(\mathcal{M}) = D^b(\mathcal{M})$, i.e. whether a localization type theorem stronger (but less explicit) than the one obtained by Schneider and Stuhler in [Sch-St1],[Sch-St2] holds. The positive answer to this question is equivalent to the claim that the Grothendieck group of $\mathcal{M}$ is generated by the classes of standard projectives.

Notice that the following unpublished result of J. Bernstein agrees with the positive answer to this question: the Grothendieck group of $\mathcal{M}$ tensored with $\mathbb{Q}$ is generated by the classes of standard projectives.

Remark 2.4. Our method also does not answer the question whether $D^+(\mathcal{M}\check{})$ or $D^+(\mathcal{M})$ is localization of $D^+(\mathcal{S}\mathfrak{h})$. (Any module has infinite right resolution by standard injectives; our procedure of constructing a complex of sheaves from such a resolution was inductive "moving from right to the left".)

3. **Main theorem.** We start the construction of the functor $\mathcal{L}$ promised in the introduction.

Let Z be a closed subset of $\overline{X}-X$.

Consider the subspace $\Gamma_c(\overline{X}-Z, j_*(\mathcal{F}))\subset\Gamma(X,\mathcal{F})$ and the subspace $\Gamma_c(\overline{X}-Z, j_*^G\mathcal{F})\subset\Gamma(\overline{X}, j_*^G\mathcal{F}) = Sm(\Gamma(X,\mathcal{F}))$. (The latter equality readily follows from compactness of $\overline{X}$.)

We now explain how to recover these subspaces from an object $\Gamma(X,\mathcal{F})\in\mathcal{M}\check{}$.

Fix a point $p\in X$. Let $a: G\times X\to X$ be the action; set $a_p(g)=a(g,p)$. Let $\mathfrak{F}_Z$ denote the filter of subsets in G generated by the sets of the form $a_p^{-1}(U)$, where U is a neighborhood of Z.

For a module $L\in\mathcal{M}\check{}$ define $L_Z = \{m\in L \mid \lim_{\mathfrak{F}}(g^{-1}(m))=0\}$, and $L_Z^{\mathfrak{sm}} = Sm(L)\cap L_Z$. (Here $\lim_{\mathfrak{F}}$ stands for the limit with respect to the filter $\mathfrak{F}$.)

Lemma 3.1. *a) The filter $\mathfrak{F}$ is invariant under right translations on G. It does not depend on the choice of a point $p\in X$.*

b) We have:

$\Gamma_c(\overline{X}-Z, j_*(\mathcal{F})) = (\Gamma(X,\mathcal{F}))_Z$

$\Gamma_c(\overline{X}-Z, j_*^G(\mathcal{F})) = (\Gamma(X,\mathcal{F}))_Z^{\mathfrak{sm}}$

Proof To prove a) we need the following well-known property of $\overline{X}$.

Fact 3.2. *Assume that a sequence of points $x_n\in X$ has a limit $x\in\overline{X}-X$. Let $y_n\in X$ be another sequence, such that $d(x_n,y_n)$ is bounded. Then $y_n\to x$ as well.*

Proof is included since no reference was found. We use Remark 5.5(2) in [BoSer]. It says the following. Fix an open polysimplex $\Delta_0\subset X$, and for any apartment $\mathfrak{A}\subset X$ containing Δ_0 let $\Xi_{\mathfrak{A}}: X\to\mathfrak{A}$ be the contraction of X to Δ_0 [BT1] 2.3.4, 7.4.19. Recall that an apartment $\mathfrak{A}$ is an affine space, and let $\overline{\mathfrak{A}}$ be its standard semisphere compactification (see below); denote by $j_{\mathfrak{A}}$ the imbedding $\mathfrak{A}\hookrightarrow\overline{\mathfrak{A}}$. Then

$\overline{X}$ is the closure of the image of the map $\prod_{\mathfrak{A}}(j_{\mathfrak{A}} \circ \Xi_{\mathfrak{A}}) : X \to \prod_{\mathfrak{A}} \overline{\mathfrak{A}}$. (The product is taken with the Tikhonov topology.) In particular two sequences x_n, y_n have the same limit in $\overline{X}$ if and only if $\lim \Xi_{\mathfrak{A}}(x_n) = \lim \Xi_{\mathfrak{A}}(y_n)$ for any $\mathfrak{A} \supset \Delta_0$. However, by [BT1] 7.4.20(ii) the map $\Xi_{\mathfrak{A}}$ does not increase distances. Hence the statement is reduced to the analogous statement about sequences in the compactified affine space $\overline{\mathfrak{A}}$, which is obvious. □

The statement a) follows from the Fact directly. b) is clear. □

3.3. For $\mathfrak{M} \in \mathcal{M}^{\check{}}$ we now define a presheaf $\widetilde{\mathcal{L}}(\mathfrak{M})$ on Y by $\Gamma(U, \widetilde{\mathcal{L}}(\mathfrak{M})) := \varprojlim_{Z} Sm(\mathfrak{M})/(\mathfrak{M}_Z \cap Sm(\mathfrak{M}))$ where Z runs over all closed subsets $Z \subset U$. We define $\mathcal{L}(\mathfrak{M})$ to be the associated sheaf.

From the definition it is clear that the stalk of $\mathcal{L}(\mathfrak{M})$ at a point $z \in Y$ equals $Sm(\mathfrak{M})/(\mathfrak{M}_{\{z\}} \cap Sm(\mathfrak{M}))$.

The desired properties of the functor $\mathcal{L}$ follow from the next

Theorem 3.4. *a) Let $z \in \overline{X} - X$ be a point. Then the functor $\mathfrak{M} \to \mathfrak{M}^{\mathfrak{sm}}_{\{z\}}$ on the category $\mathcal{M}^{\check{}}$ is exact.*

b) Let P denote the stabilizer of the point z, then P is a parabolic subgroup in G. Let r_P be the corresponding Jacquet functor (that is coinvariants with respect to the unipotent radical of P). For any $\mathfrak{M} \in \mathcal{M}^{\check{}}$ the subspace $\mathfrak{M}^{\mathfrak{sm}}_{\{z\}} \subset Sm(\mathfrak{M})$ contains the kernel of the projection $Sm(\mathfrak{M}) \to r_P(Sm(\mathfrak{M}))$. Furthermore, if $\mathfrak{M}$ lies in the category $\mathcal{R}^{\check{}}$ then the sequence

$$0 \to \mathfrak{M}^{\mathfrak{sm}}_{\{z\}} \to Sm(\mathfrak{M}) \to r_P(Sm(\mathfrak{M})) \to 0$$

is exact.

Proof occupies the rest of this section.

3.5. We abbreviate $\mathfrak{F} = \mathfrak{F}_{\{z\}}$. Our first aim is to get a more tractable description of $\mathfrak{F}$.

Fix a Levi decomposition $P = L \cdot N$, and a maximal split torus $T \subset L$. Let A be the center of L, and A^s be its maximal split subtorus.

We have vector spaces $\mathfrak{a} = \mathrm{Hom}(\underline{F^\times}, \underline{A}) \otimes \mathbb{R} = (A/A^c) \otimes \mathbb{R} = (L/L^c) \otimes \mathbb{R}$ where $A^c \subset A$ is the maximal compact subgroup and $L^c \subset L$ is the subgroup generated by all compact subgroups; and $\mathfrak{a}_T = \mathrm{Hom}(\underline{F^\times}, \underline{T}) \otimes \mathbb{R} = (T/T^c) \otimes \mathbb{R}$ where $T^c \subset T$ is the maximal compact subgroup. The imbedding $\mathfrak{a} \hookrightarrow \mathfrak{a}_T$ is obtained from the imbedding $\mathrm{Hom}(\underline{F^\times}, \underline{A}) = \mathrm{Hom}(\underline{F^\times}, \underline{A_s}) \hookrightarrow \mathrm{Hom}(\underline{F^\times}, \underline{T})$.

(Here we use the notation $\underline{H}$ for an algebraic group over F, and H for its group of F-points.) We identify $\mathrm{Hom}(\underline{F^\times}, \underline{A})$ and L/L^c with subgroups in $\mathfrak{a}$ and the groups $\mathrm{Hom}(\underline{L}, \underline{F^\times})$, $\mathrm{Hom}(\underline{A}, \underline{F^\times})$ with subgroups in the dual space $\mathfrak{a}^*$.

We fix a uniformizer $\mathfrak{p}$ in $F^\times$, and get cocompact imbeddings $\mathrm{Hom}(\underline{F^\times}, \underline{T}) \hookrightarrow T$, $\mathrm{Hom}(\underline{F^\times}, \underline{A}) \hookrightarrow A$, given by $\psi \mapsto \psi(\mathfrak{p})$. We denote the image of latter imbedding by Λ.

Let π be the projection $L \to L/L^c$.

By Λ^+ we denote the set of $\lambda \in \Lambda$ such that $\pi(\lambda) \in \mathfrak{a}^+$ where $\mathfrak{a}^+$ is the positive chamber, $\mathfrak{a}^+ = \{a \in \mathfrak{a} | r(a) < 0 \text{ if } r \text{ is a root of } A \text{ in } N\}$.

We fix notations and recall some basic facts about the geometry of X (see [BT1], [BT2], [L] for more details).

Recall that the set of maximal split tori is in bijection with apartments in X. Let $\mathfrak{A}$ be the apartment corresponding to T, and let $\overline{\mathfrak{A}} \subset \overline{X}$ be the closure of $\mathfrak{A}$.

Then $\mathfrak{A}$ is an affine space with underlying vector space $\mathfrak{a}_T$. The torus T acts on $\mathfrak{A}$ through T/T^c acting on the affine space by translations.

The closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ in $\overline{X}$ is the standard semisphere compactification of the affine space $\mathfrak{A}$. In particular $\overline{\mathfrak{A}} - \mathfrak{A}$ is the set of rays $(\mathfrak{a}_T - \{0\})/\mathbb{R}^*_+$. By our assumption the point z lies in $\overline{\mathfrak{A}} - \mathfrak{A}$ and corresponds to a ray $\rho_z \subset \mathfrak{a} \subset \mathfrak{a}_T$. We have $\rho_z \subset \mathfrak{a}^+$.

By a cone in a vector space we will mean a convex centered cone.

Definition 3.6. a) A z-good cone is an open cone $C \subset \mathfrak{a}$ such that $\rho_z \subset C$.

For a z-good cone set $\Lambda_C = \Lambda \cap \pi^{-1}(\overline{C} \cap (L/L^c))$ and $\Lambda_C^+ = \Lambda \cap \pi^{-1}(C \cap (L/L^c))$.

b) A z-good semigroup is an open compactly generated semigroup $M \subset L$ such that

i) M contains Λ_C for a z-good cone C.

ii) We have $L = \Lambda \cdot M$.

It is easy to see that an intersection of two z-good semigroups is again a z-good semigroup.

Proposition 3.7. *The filter $\mathfrak{F}$ is generated by the sets*

$$K' \cdot M \cdot a \cdot K_p,$$

where K' is an open compact subgroup in G, M is a z-good semigroup, $a \in \Lambda$ and $p \in \mathfrak{A}$.

We need some more notations and basic facts about $\overline{X}$.

Recall that $d(\cdot,\cdot)$ denotes the canonical metric on X.

For $x \in X$ denote by K_x the stabilizer of x. It is an open compact subgroup in G.

Lemma 3.8. *a) For any points $c \in X$, $z \in \overline{X} - X$ there exists a unique geodesic $[c, z)$ connecting c and z. (That is $[c, z)$ is the unique geodesic ray starting at c and having z as its limit point in $\overline{X} - X$.) For $c \in \mathfrak{A}$ the ray $[c, z)$ coincides with $\rho_z + c$.*

b) Assume that $c_1, c_2 \in X$, $z \in \overline{X} - X$ and let c_i^t be the unique point on $[c_i, z)$ at the distance t from c_i for $i = 1, 2$. Then we have

$$d(c_1^t, c_2^t) \leqslant d(c_1, c_2) \tag{6}$$

Proof a) follows from [Sch-St2], Proposition on p. 166. More precisely, in *loc. cit.* it is proven that there exists an apartment $\mathfrak{A}$ containing c and z in its closure, and that the ray $[c, z)_{\mathfrak{A}} \subset \mathfrak{A}$ of direction given by z with vertex at c does not depend on the choice of $\mathfrak{A}$.

This gives existence; to show uniqueness it remains only to check that any geodesic ray ρ lies in some apartment. Let $\mathfrak{A}$ be any apartment containing the vertex c of the ray ρ. By [BT1] 2.3.4 the set $\mathfrak{C}_t := \{g \in K_c \,|g(c'_t) \in \mathfrak{A}\}$ is nonempty for all t. Each of the sets $\mathfrak{C}_t$ is closed (and open) since it is a finite union of cosets of an open compact subgroup $K_c \cap K_{c'_t}$; also $\mathfrak{C}_{t_1} \subset \mathfrak{C}_{t_2}$ if $t_1 > t_2$ because $\mathfrak{A}$ contains the geodesic connecting any two points of $\mathfrak{A}$, [BT1] 7.4.20(iii). Since K_c is compact we have $\bigcap\limits_t \mathfrak{C}_t \neq \emptyset$. Now we can take $\mathfrak{A}' := g^{-1}(\mathfrak{A})$ for any $g \in \bigcap\limits_t \mathfrak{C}_t$.

Let us prove b). Let $x_1 = c_1, x_2, \dots, x_n = c_2$ be a sequence of points on $[c_1, c_2]$ such that $[c_1, c_2) = \cup [x_i, x_{i+1})$, and x_i, x_{i+1} lie in the closure of an open polysimplex Δ_i.

By [BT1] 7.4.18(ii) there exists an apartment $\mathfrak{A}_i$ such that $\mathfrak{A}_i \supset \Delta_i$, and $\overline{\mathfrak{A}}_i \ni z$. Thus $[x_i, z)$, $[x_{i+1}, z)$ are parallel rays in the affine space $\mathfrak{A}_i$, so $d(x_i^t, x_{i+1}^t) =$

$d(x_i, x_{i+1})$ for all t. We finally get

$$d(c_1^t, c_2^t) \leqslant \sum_{j=1}^{n-1} d(x_j^t, x_{j+1}^t) = \sum_{j=1}^{n-1} d(x_j, x_{j+1}) = d(c_1, c_2). \square$$

3.9. For any $x \in \mathfrak{A}$ the action map $\mu : K_x \times \overline{\mathfrak{A}} \to \overline{X}$ is proper, surjective and the topology on $\overline{X}$ is the quotient topology with respect to this map [BoSer], 5.4.1. Hence the filter of neighborhoods of $z \in \overline{X} - X$ is generated by images of neighborhoods of $\mu^{-1}(z)$ in $K_x \times \overline{\mathfrak{A}}$.

Fix a point $x \in \mathfrak{A}$ which lies inside an open polysimplex $\Delta_0 \subset \mathfrak{A}$. Then $K_x(z) \cap \overline{\mathfrak{A}} = \{z\}$. Indeed, if, on the contrary, $z \neq k(z) \in \overline{\mathfrak{A}}$ for some $k \in K_x$, then $k([x, z)) = [x, k(z'))$ is a ray in $\mathfrak{A}$ different from $[x, z)$; however, since k fixes the neighborhood Δ_0 of x in $\mathfrak{A}$, the intersection of the two rays with this neighborhood coincide, which is impossible.

Thus $\mu^{-1}(z) = Stab_{K_x}(z) \times \{z\} = (P \cap K_x) \times \{z\}$.

Let $P^- = L \cdot N^-$ be the parabolic opposite to P. Since x lies in an open polysimplex of $\mathfrak{A}$ we have the following triangular decomposition:

$$K_x = K_x^- \cdot K_x^0 \cdot K_x^+ \tag{7}$$

where $K_x^- = K_x \cap N^-$, $K_x^0 = K_x \cap L$, $K_x^+ = K_x \cap N^+$ (for P minimal this follows from [BT1] 7.1.4(2), and the general case is a corollary).

The fundamental system of neighborhoods of $K_x \cap P = K_x^0 \cdot K_x^+$ is formed by subgroups of the form $O = O^- \cdot K_x^0 \cdot K_x^+$ where O^- is a small open compact subgroup in N^-.

The filter of neighborhoods of z in $\overline{\mathfrak{A}}$ is generated by closures of the cones $\widetilde{C} + t$ where $\widetilde{C} \subset \mathfrak{a}_T$ is an open cone containing ρ_z, and $t \in \mathfrak{A}$ is a point.

Since K_x and $\overline{\mathfrak{A}}$ are compact it follows that the filter of neighborhoods of $(K_x \cap P) \times \{z\}$ is generated by the sets $O \times (\widetilde{C} + t)$ in the above notations. Thus the sets $O(\widetilde{C} + t)$ generate the filter of neighborhoods of z in $\overline{X}$.

Further, it is enough to take only such $\widetilde{C}$, t that $\alpha|_{\widetilde{C}} \leqslant 0$ if α is a root of T in N, and $t \in x + \widetilde{C}$. Then we have $\widetilde{C} + t \subset \widetilde{C} + x \subset \bigcup_{z', P_{z'} \subseteq P} [x, z')$. Since $K_x^+ \subset K_x \cap P'$ for any $P' \subset P$ we see that K_x^+ fixes $\widetilde{C} + t$ pointwise.

Thus the system of neighborhoods of z in X is generated by $O(\widetilde{C} + t) = O^- \cdot K_x^0 \cdot K_x^+(\widetilde{C} + t) = O^- \cdot K_x^0(\widetilde{C} + t)$ for $O, \widetilde{C}, t$ as above; we will call $O(\widetilde{C} + t)$ a *standard* neighborhood of z.

3.10. *Proof* of Proposition 3.7. Fix a z-good semigroup M, an element $a \in \Lambda_C$ for a z-good cone C, and an open subgroup $K' \subset G$. First we want to find a neighborhood U of z in X such that $a_p^{-1}(U) \subset K' \cdot M \cdot a \cdot K_p$, i.e. $U \subset K' \cdot M \cdot a(p)$.

Claim *Let π_T denote the projection from T to $T/T^c \subset \mathfrak{a}_T$. Then the semigroup $\pi_T(M \cap T)$ contains $(T/T^c) \cap \widetilde{C}$ for some open cone $\widetilde{C} \subset \mathfrak{a}_T$ containing ρ_z.*

Proof Notice that $\pi_T(M \cap T) + A/A^c = T/T^c$. Moreover, $(\pi_T(M \cap T)) \cap \mathfrak{a}$ contains $(T/T^c) \cap C = (A/A^c) \cap C$ for a z-good cone $C \subset \mathfrak{a}$. This yields the claim by the following elementary argument.

Induction in $\dim(\mathfrak{a}_T/\mathfrak{a})$ reduces the statement to the situation when $\dim(\mathfrak{a}_T/\mathfrak{a}) = 1$; in this case one can argue as follows. By the second sentence of the proof C contains a basis $v_1, \ldots, v_r$ of A/A^c such that $\sum a_i v_i \in \rho$ for some $a_i > 0$. Let v be

an element of $\pi_T(M\cap T)$ which does not lie in $\mathfrak{a}$, and take $v'\in(-v+(A/A^c))\cap\pi_T(M\cap T)$; denote the sublattice generated by $v,v_1,\dots,v_r$ by Λ'.

The subset $\{a_0v+\sum a_iv_i\,|\,a_i\geqslant 0\}\cup\{a_0v'+\sum a_iv_i\,|\,a_i\geqslant 0\}$ obviously contains an open cone C' such that $C'\supset\rho$. Any point of $\Lambda'\cap C'$ has nonnegative integral coordinates with respect to one of the bases $(v,v_1,\dots,v_r)$ or $(v',v_1,\dots,v_r)$, hence lies in $\pi_T(M\cap T)$.

On the other hand, Λ' is a sublattice of finite index in T/T^c; let $x_1,\dots,x_n\in T/T^c$ be a set of representatives for the Λ' cosets. By the first sentence of the proof we can assume that $x_i\in\pi_T(M\cap T)$.

It is clear that for a narrow enough subcone $C\subset C'$, $C\supset\rho$, we have $y\in C\cap\pi_T(M\cap T), y\notin\rho\Rightarrow y-x_i\in C'$ for all i. For some i we have $y-x_i\in\Lambda'$, so $y-x_i\in\Lambda'\cap C'\subset\pi_T(M\cap T)$, and hence $y=x_i+(y-x_i)\in\pi_T(M\cap T)$. If $y\in(T/T^c)\cap\rho$, then $y\in\pi_T(M\cap T)$ by the second sentence of the proof. □

We can assume that $\alpha|_{\widetilde{C}}\leqslant 0$ if α is a root of T in N.

Also we can find $b\in\Lambda_C$ such that $b+a+p\in x+\widetilde{C}$, so that $b+a+p+\widetilde{C}$ is point-wise fixed by K_x^+. We can furthermore assume that $M\supset b\cdot K_x^0$.

We now take $O=O^-\cdot K_x^0\cdot K_x^+$ as above such that $O^-\subset K'$. Then $K'\cdot M\cdot a(p)\supset O^-\cdot K_x^0\cdot ab\cdot(\widetilde{C}\cap(T/T^c))(p)$. So $U=O^-\cdot K_x^0(a+b+\widetilde{C}+p)=O(a+b+\widetilde{C}+p)$ is the desired neighborhood.

It remains to show that for any neighborhood U of z in $\overline{X}$ there exist M,a,K' as above such that $K'\cdot M\cdot a\cdot K_p\subset a_p^{-1}(U)$ i.e. $K'\cdot M\cdot a(p)\subset U$. We will do it with the help of another auxilary geometric statement.

For $x\in X$, $r\in\mathbb{R}$ write $B(c,r)=\{y\in X|d(y,c)<r\}$ for the ball of radius r centered at c. For a point $y\in\mathfrak{A}$ and $\lambda\in\mathbb{R}^{>0}$ we define a *sector* $S(y,\lambda)$ by $S(y,\lambda)=\cup_{t\in\mathbb{R}^{>0}}B([y,z)_t,\lambda t)$ (notations of 3.8b)).

Lemma 3.11. *a) For each neighborhood U of z in X there exists a sector $S=S(y,\lambda)$ such that $S\subset U$.*

b)Let $S\subset X$ be a sector. For any compact set $\mathfrak{C}\subset L$ and a z-good cone C there exists $a\in\Lambda_C$ such that $ag(S)\subseteq S$ for all $g\in\mathfrak{C}$.

Proof a) We can assume that $U=O(C+t)$ is a standard neighborhood.

It is obvious that $S(q,\lambda)\cap\mathfrak{A}\subset U$ for $q\in U\cap\mathfrak{A}$ and λ small enough. It will be convenient to take q inside of an open polysimplex Δ_q of the canonical polytriangulation.

By [BT1] 2.3.4, 7.4.19 any point $s\in X$ is K_q-conjugate to a unique point in $\mathfrak{A}$. Moreover, by [BT1] 7.4.20 ii), for any $k\in K_q$ and $x_1,x_2\in\mathfrak{A}$ we have $d(k(x_1),x_2)\geqslant d(x_1,x_2)$. Hence $S=S(t,\lambda)\subset K_q(S\cap\mathfrak{A})$.

We can insert q instead of x in the triangular decomposition (7). Now for λ small enough the cone we have $S(q,\lambda)\cap\mathfrak{A}\subset a_T^+ + q$, where $a_T^+=\{\lambda\in\mathfrak{a}_T|(\alpha,\lambda)\leqslant 0$ if α is a root of T in N, which implies that $S(q,\lambda)\cap\mathfrak{A}$ is point-wise fixed by K_q^+ (see above). So $S(q,\lambda)\subset K_q(S\cap\mathfrak{A})=K_q^-\cdot K_q^0(S\cap\mathfrak{A})$. It remains to notice that if we take e.g. $q=a(x)$ for appropriate $a\in\Lambda_C$ (a should be "deep enough" in $\mathfrak{a}_T^+$) then, besides $S(q,\lambda)\cap\mathfrak{A}\subset U$, we will get $K_q^-=a(K_x^-)a^{-1}\subset O^-$, $K_q^0=K_x^0$. Then $S(q,\lambda)\subset K_q^-\cdot K_q^0(S(q,\lambda)\cap\mathfrak{A})\subset O^-\cdot K_x^0(S\cap\mathfrak{A})\subset U$. a) is proven.

b) Let $\mathfrak{C}\subset L$ be a compact subset. Then $d(x,g(x))$ is bounded by a constant d_0 for $g\in\mathfrak{C}$.

Set $d_1=\sup_{t>0}\inf_{a\in\Lambda_C}d(a(x),[x,z)_t)$. It is clear that $d_1<\infty$.

Take $s \in S, g \in \mathfrak{C}$; let $t_0 \in \mathbb{R}^{>0}$, $a \in \Lambda_C$ be such that $d(s,[x,z)_{t_0}) < \lambda t_0$, $d([x,z)_{t_0}, a(x)) \leqslant d_1$. Then we have

$$d(ag(s),[x,z)_{t+t_0}) \leqslant d(ag(s), ag([x,z)_{t_0})) + d(ag([x,z)_{t_0}),[x,z)_{t+t_0}) < \lambda t_0 + d(ag(x),[x,z)_t).$$

Here in the second step we used the inequality $d(ag([x,z)_{t_0}),[x,z)_{t_0+t}) \leqslant d(ag(x),[x,z)_t)$ which is just (6) applied to the case $c_1 = ag(x)$, $c_2 = [x,z)_t$. Note that $ag(z) = z$ because $a, g \in L$, hence $ag([x,z)) = [ag(x), z)$.

But

$$d(ag(x),[x,z)_t) \leqslant d(ag(x), a(x)) + d(a(x),[x,z)_t) \leqslant d_0 + d_1.$$

Thus we get

$$d(ag(s),[x,z)_{t+t_0}) \leqslant \lambda t_0 + d_0 + d_1 < \lambda(t+t_0),$$

provided $\lambda t_0 > d_0 + d_1$. So $ag(S) \subset S$ and the Lemma is proven. □

3.12. We are now ready to finish the proof of Proposition 3.7. Assume a neighborhood U of z in $\overline{X}$ is given. We are looking for M, a, K' as above such that $K' \cdot M \cdot a(p) \subset U$.

Let $S = S(y,\lambda)$ be a sector contained in U. There exists a z-good cone $C \subset \mathfrak{a}$ and $a \in \Lambda_C$ such that $a + C + p \subset S \cap \mathfrak{A}$.

Let $\mathfrak{C}$ be a compact set generating L as a semigroup. By the previous Lemma we can find $b \in A$ such that $b \cdot \mathfrak{C}$ preserves S.

We set M to be generated by Λ_C, $K_y \cap L$ and $b \cdot \mathfrak{C}$; the element a is chosen so that $a + C + p \subset S \cap \mathfrak{A}$.

We can assume $U \cap X = O(C+t)$ is a standard neighborhood, and take $K' = O$.

Let us check that $K' \cdot M \cdot a(p) \subset U$. By the construction $\Lambda_C \cdot a(p) \subset S \cap \mathfrak{A}$. Since L commutes with Λ, and $b \cdot \mathfrak{C}$, $K_y \cap L$ preserve S we have $M \cdot a(p) \subset S$. Hence $K' \cdot M \cdot a(p) \subset K'(S) = O(S) \subset U$.

The Proposition is proven. □

3.13. We return to the proof of the Theorem.

For an open compact subset $\mathfrak{C} \subset G$ we denote by $e_{\mathfrak{C}} \in \mathcal{H}$ the constant measure on $\mathfrak{C}$ of total volume 1. We write $\cdot$ for the convolution of compactly supported measures on a group.

We can fix a small open compact subgroup $K \subset K_p$ which is normal in K_p and satisfies the triangular decomposition (7) (see 2.1 of part 2 below for a stronger statement).

Set: $L^+ = \{g \in L | g(K^-)g^{-1} \subset K^-\}$. It is readily seen that the semigroup L^+ is z-good.

We have a map $\phi : \mathcal{H}(L^+, K^0) \to \mathcal{H}(G, K)$ sending h to $e_K \cdot i_*(h) \cdot e_K$ (here $i : L^+ \hookrightarrow G$ is the embedding, and i_* denotes the direct image of a measure). We abbreviate $\mathcal{H}(L^+, K^0)$ to $\mathcal{H}_+$.

Lemma 3.14. *ϕ is a homomorphism of algebras.*

Proof see e.g. [Al], Theorem 1 (to apply *loc. cit.* as stated L^+ has to consist of semisimple elements, i.e. to be a minimal Levi, and N, N^- must be maximal unipotent subgroups. However these more restrictive conditions are not used in the proof of Theorem 1). □

Let $\mathfrak{F}_{\mathcal{H}_+}$ be the filter of subspaces in $\mathcal{H}_+$ generated by $\mathcal{H}(a \cdot M, K^0)$ where $M \subset L^+$ is a z-good semigroup containing K^0 and $a \in \Lambda_C \subset M$.

The Hecke algebra $\mathcal{H}(G)$ acts on every smooth G-module; this action is denoted by $h : m \mapsto h(m)$. For any $h \in \mathcal{H}(G)$ we denote by h^* the image of h under the involution $g \to g^{-1}$.

Proposition 3.15. *a) Fix $\mathfrak{M} \in \check{\mathcal{M}}$. Assume that $*(\mathfrak{M})$ is generated by its K-fixed vectors. Then for $f \in \mathfrak{M}^K$ the equality $\lim_{\mathfrak{F}} g^{-1}(f) = 0$ is equivalent to $\lim_{\mathfrak{F}_{\mathcal{H}_+}} \phi(h)^*(f) = 0$.*

b) The functor H_z on the category of $\mathcal{H}_+$-modules sending a module $\mathfrak{M}$ to the space $\{f \in \mathfrak{M}^* \mid \lim_{\mathfrak{F}_{\mathcal{H}_+}} \langle h(m), f\rangle = 0$ for any $m \in \mathfrak{M}\}$ (where $\mathfrak{M}^*$ is the full linear dual, and the basic field is equipped with discrete topology) is exact.

Proof a) Assume that $\lim_{\mathfrak{F}_{\mathcal{H}_+}} \phi(h)^*(f) = 0$. Then for any $m \in *(\mathfrak{M})^K$ we can find a K-invariant z-good semigroup $M \subset L^+$ and $a \in \Lambda_C$ such that

$$h \in \mathcal{H}(a \cdot M, K^0) \Longrightarrow \langle m, \phi(h)^*(f)\rangle = 0 \tag{8}$$

Hence for any finite set $m_1, .., m_l \in *(\mathfrak{M})^K$ there is some M, a as above such that (8) is satisfied for $m \in \{m_1, .., m_l\}$. Fix $m \in *(\mathfrak{M})^K$ and take the finite set to be $K_p(m)$ (recall that we assumed that K_p normalizes K, hence preserves $*(\mathfrak{M})^K$). Then we get M, a such that

$h \in \mathcal{H}(a \cdot M) \Longrightarrow \langle k(m), \phi(h)^*(f)\rangle = 0$ for $k \in K_p$. This means that $\langle m, g^{-1}(f)\rangle = 0$ for $g \in K \cdot M \cdot a \cdot K_p$ hence $\lim_{\mathfrak{F}}\langle m, g^{-1}(f)\rangle = 0$.

It remains to notice that $\{m \in *(\mathfrak{M}) \mid \lim_{\mathfrak{F}}\langle g(m), f\rangle = 0\}$ is a G-submodule in $*(\mathfrak{M})$ (because $\mathfrak{F}$ is right-translation invariant). Thus if $*(\mathfrak{M})$ is generated by its K-fixed vectors, then $\lim_{\mathfrak{F}}\langle m, g^{-1}(f)\rangle = 0$ for any $m \in *(\mathfrak{M})$, i.e. $\lim_{\mathfrak{F}} g^{-1}(f) = 0$.

Conversely, if $\lim_{\mathfrak{F}} g^{-1}(f) = 0$ then for any m there is a z-good semigroup $M \subset L^+$ and $a \in \Lambda \cap L^+$ such that $\langle m, g^{-1}(f)\rangle = 0$ for $g \in M \cdot a$. We can assume that $a \in \Lambda_C$. Any z-good semigroup contains one normalized by K^0, so we can assume also that M is normalized by K^0. Then $M \cdot K^0$ is again a z-good semigroup and $\langle m, \phi(h)^*(f)\rangle = 0$ for $h \in \mathcal{H}(a \cdot M \cdot K_0)$ provided m is K-invariant. a) is proved.

3.16. To prove b) we need an auxilary definition.

The homomorphism $\pi|_{L^+} : L^+ \to L/L^c$ defines an L/L^c-grading on $\mathcal{H}_+$.

Definition 3.17. A z-good subalgebra in $\mathcal{H}_+$ is a graded subalgebra $\mathcal{A} = \oplus_{\lambda \in L/L^c}\mathcal{A}_\lambda$ in $\mathcal{H}^+$ such that

i) For some z-good cone C all elements of the form $e_{K^0 \cdot a}$ belong to $\mathcal{A}$ for $a \in \Lambda_C$.

ii) The map $pr_* : \mathcal{A} \to \mathcal{H}(L/\Lambda, K^0)$ is surjective.

Sublemma 3.18. *Let $\mathcal{A}$ be a z-good semigroup, C be a z-good cone such that $e_{\eta \cdot K^0} \in \mathcal{A}$ for $\eta \in \Lambda_C$. Then for any $h \in \mathcal{H}(L, K^0)$ there exists $\nu \in \Lambda_C$ such that $\nu \cdot h \in \mathcal{A}$.*

Proof We can assume that h is supported in one L^c-coset λ. By condition ii) of the definition we have $pr_*(h) = \sum pr_*(h_i)$, where $h_i \in \mathcal{A}_{\lambda_i}$. Since $\mathcal{H}(L/\Lambda)$ is graded by the finite group $L/(L^c \cdot A)$, we can further assume that $\lambda_i = \lambda \mod \Lambda$. Then we get: $h = \sum(\lambda - \lambda_i)(h_i)$. Clearly for "large enough" $\nu \in \Lambda_C$ we have $\nu + \lambda - \lambda_i \in \Lambda_C$, so $(\nu)h = \sum(\nu + \lambda - \lambda_i)(h_i) \in \mathcal{A}$. □

Lemma 3.19. *a) The filter $\mathfrak{F}_{\mathcal{H}_+}$ is generated by the subspaces $a \cdot \mathcal{A}$, where $\mathcal{A}$ runs over the set of z-good subalgebras and a runs over Λ.*

b) There exist a set of z-good subalgebras $\mathcal{A}_i$ and elements $a_i \in \Lambda$, $e_{a_i \cdot K^0} \in \mathcal{A}_i$ such that:

i) $\mathfrak{F}_{\mathcal{H}^+}$ is generated by $a_i \cdot \mathcal{A}_i$
ii) Each of the algebras $\mathcal{A}_i$ is finite over its center, which has finite type.

Proof a) It is obvious that if M is a z-good semigroup, then $\mathcal{H}(M, K^0)$ is a z-good subalgebra.

On the other hand, let $\mathfrak{C}$ be a compact set generating L^c as a semigroup. Fix a z-good subalgebra $\mathcal{A}$. Since $\mathcal{H}(\mathfrak{C}, K^0)$ is finite dimensional Sublemma 3.18 gives a coweight $\nu \in \Lambda_C$ such that $(\nu) \cdot \mathcal{H}(\mathfrak{C}, K^0) \subset \mathcal{A}$.

Let M be the semigroup generated by K^0, $\mathfrak{C} \cdot \nu$ and Λ_C. Obviously M is a z-good semigroup, and $\mathcal{H}(M, K^0) \subset \mathcal{A}$. This proves a).

The proof of b) consists in constructing the Noetherian algebras $\mathcal{A}_i$ explicitly.

It is known [BD] that $\mathcal{H}(L, K^0)$ is finite over its center Z_L; and Z_L is a finitely generated commutative algebra. Let $z_1, .., z_n$ be generators of Z_L, and $h_1, .., h_l$ be the generators of $\mathcal{H}(L, K^0)$ as Z_L-module. We have: $h_i \cdot h_j = \sum z_{ij}^l h_l$ for $z_{ij}^l \in Z_L$.

Fix a rational (with respect to the lattice $L/L^c \subset \mathfrak{a}$) z-good cone $C \subset \mathfrak{a}^+$. Then the semigroup Λ_C is finitely generated.

We can find $a \in \Lambda_C^+$ such that $a \cdot z_i$, $a \cdot z_{ij}^l$ and $a \cdot h_l$ are supported inside L^+.

For such a, C define $\mathcal{A}_{C,a}$ to be the subalgebra generated by $e_{\eta \cdot K^0}$ for $\eta \in \Lambda_C$ and $a \cdot z_i$, $a \cdot z_{ij}^l$, $a \cdot h_l$.

Then the subalgebra $\mathcal{Z}_{\mathcal{A}} \subset \mathcal{A}_{C,a}$ generated by $e_{\eta \cdot K^0}$ for $\eta \in \Lambda_C$ and by $a \cdot z_i$, $a \cdot z_{ij}^l$ is central. Also $\mathcal{A}_{C,a}$ is finite over $\mathcal{Z}_{\mathcal{A}}$; namely, it is clearly generated by $a \cdot h_l$ as a $\mathcal{Z}_{\mathcal{A}}$-module. Since $\mathcal{Z}_{\mathcal{A}}$ is a finitely generated commutative algebra, it is Noetherian, so the center of $\mathcal{A}_{C,a}$ is a finite $\mathcal{Z}_{\mathcal{A}}$-module, hence itself is a finitely generated commutative algebra.

To finish the proof it is enough, in view of a), to check that every z-good subalgebra contains a subalgebra of the form $\mathcal{A}_{C,a}$. For this let $\mathcal{A}$ be a z-good algebra, and C be a z-good cone such that $e_{\lambda \cdot K^0} \in \mathcal{A}$ for $\lambda \in \Lambda_C$. Then Sublemma 3.18 guarantees existence of $a \in \Lambda_C$ such that $a \cdot z_i$, $a \cdot z_{ij}^l$, $a \cdot h_l \in \mathcal{A}$, i.e. such that $\mathcal{A}_{C,a} \subset \mathcal{A}$. $\square$

Let us finish the proof of the Proposition.

Fix $\mathcal{A} \in \{\mathcal{A}_i\}$. We can pick $\mathcal{A}_i$, a_i in the Lemma so that $\mathcal{A}_i \subset \mathcal{A}$.

The filter of subspaces in $\mathcal{A}$ generated by $a_i \mathcal{A}_i$ is invariant under both left and right multiplication by an element $h \in \mathcal{A}$, in the sense that for any i and $h \in \mathcal{A}$ there exists j such that $h \cdot a_i \mathcal{A}_i \subset a_j \mathcal{A}_j$, $a_i \mathcal{A}_i \cdot h \subset a_j \mathcal{A}_j$. Thus the space $\varinjlim (\mathcal{A}/a_i \mathcal{A}_i)^*$ carries a natural $\mathcal{A}$-module structure. Moreover, for any $\mathfrak{M} \in \widetilde{\mathcal{M}}$ we have

$$
\begin{aligned}
H_z(\mathfrak{M}) = \{ f \in \mathfrak{M}^* \mid \forall m \in \mathfrak{M} \ \varinjlim_{\mathcal{F}_{\mathcal{H}_+}} \langle h(m), f \rangle = 0 \} = \\
\{ f \in \mathfrak{M}^* \mid \forall m \in \mathfrak{M} \ \exists \ z\text{-good } \mathcal{A},\ a \in \Lambda \ \forall h \in a\mathcal{A} : \ \langle h(m), f \rangle = 0 \ \} = \\
\{ f \in \mathfrak{M}^* \mid \forall m \in \mathfrak{M} \ \exists i \ \forall h \in a_i \mathcal{A}_i : \langle h(m), f \rangle = 0 \ \} = \\
\operatorname{Hom}(\mathfrak{M}, \varinjlim (\mathcal{A}/a_i \mathcal{A}_i)^*) \quad (9)
\end{aligned}
$$

So to prove the Proposition we have only to show that $\varinjlim (\mathcal{A}/a_i \mathcal{A}_i)^*$ is an injective $\mathcal{A}$-module.

It is not hard to see that for fixed i the filter of subspaces in $\mathcal{A}$ generated by $a_i^n \mathcal{A}_i$, $n \in \mathbb{Z}^{>0}$ is also invariant under left and right multiplication by $h \in \mathcal{A}$, thus we have an $\mathcal{A}$-module $\mathcal{A}_{z,i}^* \overset{\text{def}}{=} \varinjlim_n (\mathcal{A}/a_i^n \mathcal{A}_i)^*$.

Notice that $\varinjlim (\mathcal{A}/a_i \mathcal{A}_i)^* = \varinjlim_i (\varinjlim_n (\mathcal{A}/a_i^n \mathcal{A}_i)^*) = \varinjlim_i \mathcal{A}_{z,i}^*$ (in fact we can assume that $\forall i,\ n \ \exists j \mid \mathcal{A}_i = \mathcal{A}_j$, $a_j = a_i^n$).

Since $\mathcal{A}$ is Noetherian the direct limit of injective $\mathcal{A}$-modules is injective. Hence we have only to show that $\mathcal{A}_{z,i}^*$ is an injective $\mathcal{A}$-module. For any $\mathcal{A}$-module M we have

$$\begin{aligned}\mathrm{Hom}(M, \mathcal{A}_{z,i}^*) = \{f \in M^* \mid \forall m \in M \; \exists n \in \mathbb{Z}^{>0} : \; \langle f, a_i^n \mathcal{A}_i(m)\rangle = 0\} \\ = \mathrm{Hom}_{\mathcal{A}_i}(M, \varinjlim{}_n (\mathcal{A}_i/a_i^n\mathcal{A}_i)^*). \quad (10)\end{aligned}$$

So we need only to show that $\varinjlim{}_n(\mathcal{A}_i/a_i^n\mathcal{A}_i)^*$ is an injective $\mathcal{A}_i$-module.

Since $\mathcal{A}_i$ is Noetherian, this is equivalent to saying that the functor $\mathrm{Hom}_{\mathcal{A}_i}(\_, \varinjlim{}_n(\mathcal{A}_i/a_i^n\mathcal{A}_i)^*)$ is exact on the category of *finitely generated* $\mathcal{A}_i$-modules.

On this category the functor coincides with the functor $M \to \varinjlim{}_n(M/a^n)^*$, and its exactness is guaranteed by the standard Artin-Rees lemma because of 3.19b), property ii).

The Proposition is proved. □

3.20. We are ready at last to finish the proof of Theorem 3.4.

Proof of a): The statement is equivalent to saying that the functor $M \to (*(M))_{\{z\}}^{\mathfrak{sm}}$ on the category $\mathcal{M}$ is exact. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{M}$. Let K be as above (a compact open subgroup with triangular decomposition (7)), and assume that M_i is generated by its K-fixed vectors for $i = 1, 2, 3$. Then by Proposition 3.15a):

$$(*(M_i))_{\{z\}}^K = \{f \in (M_i^K)^* \mid \lim_{\mathfrak{F}_{\mathcal{H}_+}} \phi(h)^*(f) = 0\}.$$

Hence the sequence

$$0 \to (*(M_3))_{\{z\}}^K \to (*(M_2))_{\{z\}}^K \to (*(M_1))_{\{z\}}^K \to 0$$

is exact. The statement a) of 3.4 follows.

To prove b) it is convenient to use the following

Lemma 3.21. *Let $M \in \widetilde{\mathcal{M}}$ be a module, K as above. For an element $m \in M^K$ the following are equivalent:*

i) m lies in the kernel of the projection $M \to r_P(M)$

ii) $e_{Ka^{-1}K}(m) = 0$ for some $a \in \Lambda^+$.

Proof is well-known; see e.g. [BZ], Lemma 3.22 for the case $G = GL(n)$, or [BR], Claim 15.4 for the (absolutely analogous) general case. □

Now for $\mathfrak{M} \in \mathcal{M}^{\check{}}$ assume that $m \in (\mathfrak{M})^K$ lies in the kernel of the projection $Sm(\mathfrak{M}) \to r_P(Sm(\mathfrak{M}))$. Then for some $a \in \Lambda^+$ we have:

$\phi(e_{aK^0})^*(m) = e_{Ka^{-1}K}(m) = 0 \Longrightarrow \phi(e_{aK^0} \cdot \mathcal{H}_+)^*(m) = 0$

i.e. $m \in \mathfrak{M}_{\{z\}}$ by Proposition 3.15a).

Assume now that $\mathfrak{M} \in \mathcal{R}^{\check{}}$, and $m \in \mathfrak{M}_{\{z\}}^K$. For $v \in *(\mathfrak{M})^K$ we have $\lim_{\mathfrak{F}_{\mathcal{H}_+}} \langle \phi(h)^*(m), v\rangle = 0$ (by Proposition 3.15a)) hence there exists $a \in \Lambda^+$ such that $\langle e_{Ka^{-1}K}(m), v\rangle = \langle \phi(e_{aK^0})^*(m), v\rangle = 0$. Since $*(\mathfrak{M})^K$ is finite dimensional we can find one $a \in \Lambda^+$ such that $\langle e_{Ka^{-1}K}(m), v\rangle = 0$ for all $v \in *(\mathfrak{M})^K$. But this means that $e_{Ka^{-1}K}(m) = 0$. By the last Lemma this finishes the proof of Theorem 3.4. □

Theorem 3.22. *For $M \in \mathcal{M}$ let $L(M) \in Sh_G(Y)$ be the unique quotient of the constant sheaf $\underline{M}$ such that for a parabolic P and $p \in Y$ with $Stab(p) = P$ (i.e. $p \in \Delta_P$) the stalk of $L(M)$ at p is the Jacquet functor $r_P(M)$.*

The functor $\mathcal{L} : \mathcal{M}^{\check{}} \to Sh_G(Y)$ satisfies:

i) $\mathcal{L} \circ \Gamma \cong i^* \circ j_*^G$

ii) $\mathcal{L}$ is exact

iii) We have a functorial surjection $L(Sm(M)) \to \mathcal{L}(M)$; it is an isomorphism provided $M \in \mathcal{R}^{\check{}}$.

Proof Property i) follows from the Lemma 3.1. Properties ii) and iii) follow from Theorem 3.4. □

4. **Application to representation theory: dualities.** Recall the definition of *Deligne-Lusztig* duality (originally defined in the context of groups over finite field in [DL]). For two parabolic subgroups $P_2 \subset P_1 \subset G$ we have the canonical morphism $d_{P_1,P_2} : i_{P_1} \circ r_{P_1} \to i_{P_2} \circ r_{P_2}$. For any $M \in \mathcal{M}$ define the complex $DL(M)$ as follows. Set $(DL(M))^i = \bigoplus_{P \in S_i} i_P \circ r_P(M)$ where S_i is the set of standard parabolics of corank i; the differential $d_i := \sum_{P_1 \in S_i, P_2 \in S_{i+1}} \pm d_{P_1,P_2}$ (definition of signs is straightforward and can be found in [DL]). Obviously DL extends to an exact functor $Kom(\mathcal{M}) \to Kom(\mathcal{M})$ and defines a functor on the derived category.

We will need the following standard Lemma

Lemma 4.1. *$D^+(\mathcal{R})$ (respectively $D^b(\mathcal{R})$) is equivalent to the full subcategory of $D^+(\widetilde{\mathcal{M}})$ (respectively $D^b(\widetilde{\mathcal{M}})$) consisting of complexes with admissible cohomology.*

Proof Let $\tilde{\mathcal{R}} \subset \widetilde{\mathcal{M}}$ be the full subcategory consisting of direct limits of admissible modules. By [BD] $\mathcal{H}$ is finite over its center $\mathcal{Z}$ which is locally of finite type. Let $I \subset \mathcal{Z}$ be an ideal of finite codimension. Then the module $\mathcal{H}/I^n$ is admissible for every n, and by the Artin-Rees Lemma the functor $M \to \mathrm{Hom}_{\mathcal{H}}(M, \varinjlim_n (\mathcal{H}/I^n)^{\check{}}) = \mathrm{Hom}_{\mathcal{Z}}(M, \varinjlim_n (\mathcal{Z}/I^n)^{\check{}})$ is exact, i.e. the module $\check{\mathcal{H}}_I := \varinjlim_n (\mathcal{H}/I^n)^{\check{}}$ is injective. It is clear that any module in $\tilde{\mathcal{R}}$ injects into $\check{\mathcal{H}}_I$ for some I. Hence by [H], Proposition I.4.8 the tautological functors $D^b(\tilde{\mathcal{R}}) \to D^b(\widetilde{\mathcal{M}})$, $D^+(\tilde{\mathcal{R}}) \to D^+(\widetilde{\mathcal{M}})$ induce equivalences with the full subcategories of objects, whose cohomology lie in $\tilde{\mathcal{R}}$.

To finish the argument notice that for any complex $C \in Kom^b(\tilde{\mathcal{R}})$ (respectively $C^\cdot \in Kom^+(\tilde{\mathcal{R}})$) it is easy to represent C as a direct limit of its quasiisomorphic subcomplexes $C_i \in Kom^b(\mathcal{R})$ (respectively $C_i \in Kom^+(\mathcal{R})$). It follows that $D^{b,+}(\mathcal{R})$ is equivalent to the full subcategory of $D^{b,+}(\tilde{\mathcal{R}})$ whose objects have admissible cohomology. □

We are now in the position to prove

Theorem 4.2. *a) There exists a canonical morphism of functors from $D^b(\mathcal{M})$ to $D^b(\widetilde{\mathcal{M}})$*

$$DL \circ \check{} \to D_h. \tag{11}$$

b) Restriction of (11) to the full subcategory $D^b(\mathcal{R}) \subset D^b(\widetilde{\mathcal{M}})$ gives an isomorphism of functors from $D^b(\mathcal{R})$ to itself

$$D_h \cong DL \circ \check{} \tag{12}$$

Remark 4.3. Isomorphism (12), as well as its generalization (16), is due to J. Bernstein (unpublished). An attempt to interpret it through the geometry of $\overline{X}$ was the starting point for this part of the work.

Recall that another construction of (12) using compactified building appears in [Sch-St2].

Remark 4.4. Theorem 4.2 is the main step in the proof of Zelevinsky's conjecture [Z], which says that the map on the Grothendieck group of the category of admissible representations induced by DL sends the class of an irreducible representation to the class of an irreducible representation (up to a sign). The conjecture is proved in [B], [Sch-St2].

Proof of the Theorem. For $M \in \widetilde{\mathcal{M}}$ denote by $\mathcal{DL}(M) \in Sh_G(\overline{X})$ the kernel of the canonical surjection $\underline{M}_{\overline{X}} \to L(M)$. Then $\mathcal{DL}$ is an exact functor from $\widetilde{\mathcal{M}}$ to $Sh_G(\overline{X})$.

For an object $I \in \mathfrak{I}_G$ we have the canonical surjections $\mathsf{pr}_I : \underline{Sm(\Gamma(I))}_{\overline{X}} \to j_*^G(I)$ and $L(\Gamma(I)) \to \mathcal{L}(\Gamma(I)) = i^* j_*^G(I)$ (3.22(iii)), which yield a morphism of functors $s : \mathcal{DL} \circ Sm \circ \Gamma(I) \to Ker(j_*^G(I) \to i^* j_*^G(I)) = j_!(I)$.

Clearly $\mathcal{DL}$, s extend to a functor (respectively, a morphism of functors) on bounded derived categories; we will denote these extensions by the same symbols.

According to [Schn1], Lemma 1 for any locally compact G-space Z the category $Sh_G(Z)$ has enough injectives, so the derived functor of a left-exact functor on $Sh_G(Z)$ is defined. Suppose further that Z is compact; then one checks immediately that for $\mathcal{F} \in Sh_G(Z)$ the G-module $\Gamma(\mathcal{F})$ is smooth. Thus in this case we have the derived functor $R\Gamma : D^+(Sh_G(Z)) \to \widetilde{\mathcal{M}}$, which commutes with the forgetful functors $For_{sh} : Sh_G(Z) \to Sh(Z)$, $For_{mod} : \widetilde{\mathcal{M}} \to Vect_k$. By Corollary 3 in [Schn1] the forgetful functor $Sh_G(X) \to Sh(X)$ sends injective equivariant sheaves into c-soft sheaves; if Z is compact they are adapted to Γ, hence we have $For_{mod} \circ R\Gamma \cong R\Gamma \circ For_{sh}$.

We will be interested in the case $Z = \overline{X}$; then all of the above properties are satisfied.

Lemma 4.5. *$R\Gamma(\mathcal{DL}(M))$ is canonically isomorphic to $DL(R\Gamma(M))$ for any $M \in D^b(\widetilde{\mathcal{M}})$.*

Proof Recall that $\overline{X} - X$ is the spherical building of G, that is the realization of the simplicial topological space, whose space of nondegenerate n-simplices is the disjoint union of partial flag varieties G/P, where $\mathrm{corank}(P) = n+1$; the structure maps come from the canonical projections $\pi_{P_1,P_2} : G/P_1 \to G/P_2$ for $P_1 \subset G/P_2$. The sheaf $L(M)$ is the realization of a simplicial sheaf on this space; such a sheaf has a canonical resolution by acyclic sheaves.

More precisely, for a set s of simple roots (of the system of nonmultipliable roots of G) let $\mathfrak{P}(s)$ denote the corresponding conjugacy class of parabolics (so s is the set of simple roots which appear in the nilradical of $P \in \mathfrak{P}(s)$). Let $S(s) = \bigcup\limits_{P \in \mathfrak{P}(s)}$ be the corresponding stratum of $\overline{X} - X$, and $U(s)$ be the union of the stars of all simplices Δ_P, $P \in s$. Then we have $U(s_1 \cup s_2) = U(s_1) \cap U(s_2)$; thus $\{U_s \mid \#(s) = 1\}$ form an open covering of $\overline{X} - X$, and the corresponding Čech resolution of $L(M)$

has the form

$$0 \to \bigoplus_{|s|=1} j_s^* j_{s_*}(L(M)) \to \cdots \to \bigoplus_{|s|=\mathrm{rank}(G)} j_s^* j_{s_*}(L(M)) \to 0, \tag{13}$$

where j_s denotes the imbedding $U_s \hookrightarrow \overline{X}$. Hence for any $M \in \widetilde{\mathcal{M}}$ the complex

$$0 \to \underline{M}_{\overline{X}} \to \bigoplus_{|s|=1} j_s^* j_{s_*}(L(M)) \to \cdots \to \bigoplus_{|s|=\mathrm{rank}(G)} j_s^* j_{s_*}(L(M)) \to 0 \tag{14}$$

is a resolution for $\mathcal{DL}(M)$. Moreover, there exists a canonical retraction of U_s on $S(s)$, hence $H^i(U_s, j_s^*(L(M))) = H^i(S(s), L(M)|_{S(s)}) = 0$ for $i > 0$, and $H^0(U_s, j_s^*(L(M))) = H^0(S(s), L(M)|_{S(s)}) = i_P^G \circ r_P^G(M)$. Of course, $H^i(\underline{M}_{\overline{X}}) = 0$ for $i > 0$ and $H^0(\underline{M}_{\overline{X}}) = M$ since $\overline{X}$ is contractible. Thus the terms of the complex of global section of (14) are identified with that of $DL(M)$; it is easy to see that the differentials are also the same.

It is also evident how to extend the assignment $M \mapsto$ (14) to the definition of an exact functor $\check{C}\mathcal{DL}$ from $Kom(\widetilde{\mathcal{M}})$ to the complexes of Γ-adapted objects of $Sh_G(\overline{X})$, together with a canonical quasiisomorphism $\mathcal{DL} \to \check{C}\mathcal{DL}$, and a canonical isomorphism $\Gamma \circ \check{C}\mathcal{DL} \cong DL$. □

We can now construct the morphism (11). For $\mathcal{F} \in D^b(\mathcal{S}\mathfrak{h})$ we have functorial morphisms

$$DL(R\Gamma_c(\mathcal{F})\check{}) \overset{1.3a)}{\cong} DL(Sm \circ R\Gamma(\mathbb{V}\mathcal{F})) \overset{4.5}{\cong} R\Gamma(\mathcal{DL}(Sm \circ R\Gamma(\mathbb{V}\mathcal{F})))$$
$$\xrightarrow{R\Gamma(s)} R\Gamma(j_!(\mathbb{V}\mathcal{F})) = R\Gamma_c(\mathbb{V}\mathcal{F}) \overset{1.3b)}{\cong} D_h(R\Gamma_c(\mathcal{F})). \tag{15}$$

Hence by the second statement of Proposition 2.1 we get the desired morphism (11) on the full subcategory $D^0(\mathcal{M})$. a) of the Theorem now follows from the evident

Claim 4.6. *a) Let $\mathcal{A}$, $\mathcal{B}$ be additive categories, and $\mathcal{C}$ be a full subcategory of $\mathcal{A}$. Let F_1, $F_2 : \mathcal{A} \to \mathcal{B}$ be additive functors, and $s : F_1|_{\mathcal{C}} \to F_2|_{\mathcal{C}}$ be a transformation. Assume that every object of $\mathcal{A}$ is isomorphic to a direct summand in an object of $\mathcal{C}$. Then s extends in a unique way to a transformation of functors $F_1 \to F_2$.*

b) Any object of $D^b(\mathcal{M})$ is a direct summand in an object of $D^0(\mathcal{M})$.

Proof Take $A, A' \in Ob(\mathcal{A})$ such that $A \oplus A' \in Ob(\mathcal{C})$. By the definition of a morphism of functors we have $s_{A' \oplus A''} \circ (id \oplus 0) = id \oplus 0$, $s_{A' \oplus A''} \circ (0 \oplus id) = 0 \oplus id$, hence s preserves the direct sum decomposition, i.e. $s_{A \oplus A'} = s_A \oplus s'_A$. If $A'' \in Ob(\mathcal{A})$ is another object such that $A \oplus A'' \in Ob(\mathcal{C})$, then considering the morphism $id \oplus 0 : A \oplus A' \to A \oplus A''$ we see that s_A does not depend on the choice of A'. This proves a).

To prove b) recall that $\mathcal{M}$ has enough projectives and finite homological dimension ([BR], Theorem 38, see also e.g. [Vig], Proposition 37), so any object of $D^b(\mathcal{M})$ is represented by a finite complex of projective modules. Since standard projectives form a set of projective generators in $\mathcal{M}$, we can add to such a complex a complex with zero differential to get a complex of standard projectives. □

For part b) of the Theorem we need another

Lemma 4.7. *For $I \in \mathfrak{I}_G$ the cohomology $H^i(\overline{X}, j_*^G(I))$ vanish for $i > 0$.*

Sublemma 4.8. *Let $\Delta \subset X$ be a polysimplex of the canonical polytriangulation, $x \in \Delta$ be a point, and R be a smooth finite dimensional representation of $Stab(R) = K_x$. Let ι_x be the imbedding $\iota_x : G(x) \hookrightarrow X$, where $G(x)$ is the G-orbit of x, and set $\mathcal{F}_{x,R} = \iota_{x*}\iota_x^*(\mathcal{F}_{\Delta,R})$ (notations of 1.1). Consider the canonical arrow $\mathcal{F}_{\Delta,R} \to \mathcal{F}_{x,R}$, and the induced morphism $f : j_*^G(\mathcal{F}_{\Delta,R}) \to j_*^G(\mathcal{F}_{x,R})$. Then $i^*(f)$ and $R\Gamma(f)$ are isomorphisms.*

Proof of the Sublemma. Let $z \in \overline{X} - X$ be a point. Let $K_i \subset G$ be compact open subgroups forming a fundamental system of neighborhoods of 1; then there exists a basis $\{U_i\}$ of the filter of neigborhoods of z such that $U_i \cap X$ is a K_i-invariant closed simplicial subset. Indeed, if $\mathfrak{A}$ is an apartment containing z in its closure, then it is clear that one can find a basis C_i of the filter of neigborhoods of z in $\overline{\mathfrak{A}}$ such that $C_i \cap \mathfrak{A}$ is closed and simplicial; then by 3.9 we get the desired basis of the filter setting $U_i = K_i(C_i)$.

Now we see that $\Gamma(U_i, \mathcal{F}_{\Delta,R}) \cong \prod\limits_{\Delta' \in G(\Delta), \Delta' \subset U_i} R \widetilde{\longrightarrow} \Gamma(U_i, \mathcal{F}_{x,R}) \cong \prod\limits_{G(x) \cap U_i} R$. Hence $j_*^G(\mathcal{F}_{\Delta,R})|_z = \varinjlim_i \Gamma(U_i \cap X, \mathcal{F}_{\Delta,R})^{K_i} \widetilde{\longrightarrow} j_*^G(\mathcal{F}_{x,R}) = \varinjlim_i \Gamma(U_i \cap X, \mathcal{F}_{x,R})^{K_i}|_z$ (here we write $?|_z$ for a stalk of a sheaf $?$ at a point z). This proves the first statement.

It implies that f is a surjection of sheaves, and the kernel of f is (noncanonically) isomorphic to $\underline{R}_{G(\overline{\Delta}-x)}$ extended by 0 to $\overline{X}$. Since $H_c^i(\overline{\Delta} - x) = 0$ for all i the latter sheaf is acyclic, so we get the second statement. □

Proof of the Lemma. $\mathcal{F}_{x,R}$ is an injective object in $Sh_G(X)$; hence $j_*^G(\mathcal{F}_{x,R})$ is an injective object of $Sh_G(\overline{X})$. Hence by [Schn1], Corollary 3 it is a c-soft sheaf; since $\overline{X}$ is compact this implies $H^{>0}(\overline{X}, j_*^G(\mathcal{F}_{x,R})) = 0$. Thus Lemma follows from the Sublemma. □

4.9. *Proof* of Theorem 4.2b). To prove b) we must check the following. Suppose that $\mathcal{F} \in D^b(\mathcal{S}\mathfrak{h})$ is such that $R\Gamma_c(\mathcal{F}) \cong M_1 \oplus M_2$ where $M_1 \in D^b(\mathcal{R}) \subset D^b(\mathcal{M})$, $M_2 \in D^b(\mathcal{M})$. Then we have $R\Gamma(s) = S_1 \oplus S_2$, where $S_i : DL(M_i\check{}) \to D_h(M_i)$ for $i = 1, 2$, and S_1 is an isomorphism. We have $R\Gamma(s) = S_1 \oplus S_2$ for some S_1, S_2 by Claim 4.6. Now recall that for $I \in \mathfrak{I}_G$ the morphism $s : \mathcal{DL}(I) \to j_*^G(I)$ includes in a functorial morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{DL} & \longrightarrow & \underline{Sm(\Gamma(I))} & \longrightarrow & L(Sm(\Gamma(I))) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle s} & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & j_!(I) & \longrightarrow & j_*^G(I) & \longrightarrow & i^*j_*^G(I) \overset{3.22}{=} \mathcal{L}(Sm \circ \Gamma(I)) & \longrightarrow & 0 \end{array}$$

Thus we have a functorial morphism of distinguished triangles in $D^b(\widetilde{\mathcal{M}})$

$$\begin{array}{ccccc} DL(Sm \circ R\Gamma(\mathcal{F})) & \longrightarrow & Sm \circ R\Gamma(\mathcal{F}) & \longrightarrow & R\Gamma(L(Sm \circ R\Gamma(\mathcal{F}))) \\ \Big\downarrow{\scriptstyle R\Gamma(s)} & & \Big\downarrow & & \Big\downarrow \\ R\Gamma(j_!(\mathcal{F})) = R\Gamma_c(\mathcal{F}) & \longrightarrow & R\Gamma(Rj_*^G(\mathcal{F})) & \longrightarrow & R\Gamma(\mathcal{L}(Sm \circ R\Gamma(\mathcal{F}))) \end{array}$$

The second vertical map is an isomorphism by Lemma 4.7.

Also it is clear that decomposition $R\Gamma_c(\mathcal{F}) \cong M_1 \oplus M_2$ splits the whole diagram into the direct sum of two. The condition $M_1 \in D^b(\mathcal{R})$ implies that the right vertical arrow in the first summand is an isomorphism by Theorem 3.22 ii), iii).

Hence S_1 is also an isomorphism. By Lemma 4.1 the quasiisomorphism (12) in the category $D^b(\widetilde{\mathcal{M}})$ yields the corresponding quasiisomorphism in $D^b(\mathcal{R})$. □

Remark 4.10. We could have used the Schneider-Stuhler "localization" Theorem [Sch-St2] instead of our Proposition 2.1 in order to deduce the isomorphism (12) from Theorem 3.22.

4.11. We finish the chapter with a generalization of the last Theorem which connects homological and Deligne-Lusztig dualities on $D^b(\mathcal{M})$ (not only on complexes with admissible cohomology).

Recall that the Bernstein center $\mathcal{Z}$ is by the definition the algebra of endomorphisms of identity functor on $\mathcal{M}$; so it is a commutative algebra acting on every module $M \in \mathcal{M}$ commuting with the G-action. It is known [BD] that $\mathcal{Z}$ is an infinite product of finitely generated commutative algebras; in fact each factor in this decomposition is isomorphic to algebra of functions on an algebraic torus of dimension $\leqslant \mathrm{rank}(G)$ invariant under the action of a finite group.

We also recall that the strong admissibility Theorem [BD] asserts that for any $M \in \mathcal{M}$ and any open compact subgroup $K \subset G$ the space of K-invariants M^K is a finite $\mathcal{Z}$-module supported on a finite number of components of $Spec(\mathcal{Z})$. If M is a G-module, and V is a $\mathcal{Z}$-module then the space $Hom_{\mathcal{Z}}(M, V)$ is naturally a G-module; moreover, from the strong admissiblity Theorem it follows that $Sm(Hom_{\mathcal{Z}}(M, V)) \in \mathcal{M}$ if $M \in \mathcal{M}$ and V is locally finitely generated. Since $\mathcal{M}$ has enough projectives we can derive the bifunctor $(M, V) \mapsto Sm(Hom_{\mathcal{Z}}(M, V))$ in both arguments to get a bifunctor $D^-(\mathcal{M})^{opp} \times D^+(\mathcal{Z} - mod) \to D^+(\mathcal{M})$, where $\mathcal{Z} - mod$ is the category of locally finitely generated $\mathcal{Z}$-modules. Let $D \in D^b(\mathcal{Z} - mod)$ be the Grothendieck dualizing complex.

We denote by D_{Gr} the Grothendieck-Serre duality functor $RHom(\_, D)$; we have $D_{Gr} : D^b(\mathcal{M}) \to D^+(\mathcal{M})$

Note that for a finite dimensional module V over a commutative algebra we have $D_{Gr}(V) \cong V^*$ canonically. It follows that $D_{Gr}|_{D^b(\mathcal{R})} \cong \check{}$.

Remark 4.12. The quotient of a smooth variety by an action of a finite group is known to be Cohen-Macaulay, hence the dualizing complex D is concentrated in one homological dimension. Also, for such a variety the Grothendieck-Serre duality functor sends the bounded derived category $D^b(Coh)$ into itself, where Coh stands for coherent sheaves. It follows that D_{Gr} sends $D^b(\mathcal{M})$ to itself. We will not use these facts below; the latter one follows also from the next Theorem.

Theorem 4.13. *We have a canonical isomorphism of functors on* $D^b(\mathcal{M})$

$$D_h \cong DL \circ D_{Gr} \tag{16}$$

In order to prove the Theorem we need to extend the statement 4.2 to the following set-up. Let B be a commutative unital k-algebra of finite type. Let $\widetilde{\mathcal{M}}_B$ be the category of smooth $G \times B$ modules, i.e. B-modules equipped with a smooth G-action; in other words $\widetilde{\mathcal{M}}_B$ is a category of modules over the algebra $\mathcal{H}_B := \mathcal{H} \otimes B$. The category $\mathcal{M}_B \subset \widetilde{\mathcal{M}}_B$ consists of all finitely generated modules. We also define $\widetilde{\mathcal{M}\check{}}_B$ to be the category of complete topological $G \times B$ modules having a basis of neighbourhoods of 0 consisting of B-submodules such that the quotient is a finite B-module; the subcategory $\mathcal{M}\check{}_B \subset \widetilde{\mathcal{M}\check{}}_B$ consists of modules which are topologically finitely generated.

We call a module $M \in \mathcal{M}_B$ or $M \in \mathcal{M}\check{}_B$ admissible if the space of K-invariants in M is a finite B-module for any open compact subgroup $K \subset G$. We denote by $\mathcal{R}_B \subset \mathcal{M}_B$ and $\mathcal{R}\check{}_B \subset \mathcal{M}\check{}_B$ the (full) subcategories of admissible modules.

Lemma 4.14. *The category $D^+(\mathcal{R}_B)$ (respectively $D^b(\mathcal{R}_B)$) is equivalent to the full subcategory of $D^+(\widetilde{\mathcal{M}}_B)$ (respectively $D^b(\widetilde{\mathcal{M}}_B)$) consisting of complexes with admissible cohomology.*

Proof is parallel to that of 4.1. □

For $V \in B-mod$ we have a contravariant functor $Hom_B(\_, V) : \mathcal{M}_B \to \mathcal{M}\check{}_B$. We can derive it in both arguments to get a bifunctor $D^-(\mathcal{M}_B) \times D^+(B-mod) \to D^+(\widetilde{\mathcal{M}\check{}}_B)$.

We have Deligne-Lusztig functors $DL : D^{b,\pm}(\mathcal{M}_B) \to D^{b,\pm}(\mathcal{M}_B)$; the definition remains the same.

Proposition 4.15. *For any $V \in D^b(B-mod)$ there exists a canonical isomorphism of functors $D^b(\mathcal{R}_B) \to D^+(\mathcal{R}_B)$*

$$RHom_{G\times B}(\_, \mathcal{H} \otimes V)|_{\mathcal{R}_B} \cong DL \circ RHom_B(\_, V)$$

Proof is parallel to the proof of 4.2. The only difference is that one should consider G-equivariant sheaves of B-modules instead of G-equivariant sheaves of k-vector spaces, and replace the Verdier dualizing sheaf by the tensor product of Verdier dualizing sheaf (with coefficients in k) and $V \in D^+(B-mod)$.□

4.16. By the definition we have imbedding of a full subcategory $I : \mathcal{M} \hookrightarrow \mathcal{M}_{\mathcal{Z}}$. By the strong admissibility Theorem its image actually lies in $\mathcal{R}_{\mathcal{Z}}$. So to finish the proof of the Theorem it is enough to prove the following standard fact.

Proposition 4.17. *We have a functorial isomorphism*

$$RHom_{G\times\mathcal{Z}}(I \otimes Id(M), N \otimes D) \cong I(RHom_G(M, N))$$

where $M \in D^-(\mathcal{M})$; N is a bounded below complex of finitely generated smooth G-bimodules; $D \in D^b(\mathcal{Z} - mod)$ is the dualizing complex. Here both sides lie in $D^+(\widetilde{\mathcal{M}}_{\mathcal{Z}})$.

Proof We have an isomorphism

$$Hom_{G\times\mathcal{Z}}(I(M), ?) = Hom_G(M, Hom_{G\times\mathcal{Z}}(I(\mathcal{H}), ?)),$$

where the action of G on $Hom_{G\times\mathcal{Z}}(I(\mathcal{H}), ?)$ comes from the right G-action on $\mathcal{H}$; and also:

$$Hom_{G\times\mathcal{Z}}(I(\mathcal{H}), ?) = Hom_{\mathcal{Z}\otimes\mathcal{Z}}(\mathcal{Z}, ?)$$

where the first of the two $\mathcal{Z}$-actions on ? comes from the G-action on ?, and G acts on $Hom_{\mathcal{Z}\otimes\mathcal{Z}}(\mathcal{Z}, ?)$ via its action on ?. Since $Hom_{\mathcal{Z}\otimes\mathcal{Z}}(\mathcal{Z}, ?)$ is injective provided ? is injective, we get the corresponding isomorphisms of derived functors.

We have the standard quasiisomorphism in the derived category of $\mathcal{Z}$-modules

$$RHom_{\mathcal{Z}\otimes\mathcal{Z}}(\mathcal{Z}, N \otimes D) = i^! \circ pr_1^!(N) \cong (pr_1 \circ i)^! N \cong N$$

where i is the diagonal embedding $i : Spec(\mathcal{Z}) \to Spec(\mathcal{Z}\otimes\mathcal{Z})$, $pr_1 : Spec(\mathcal{Z}\otimes\mathcal{Z}) \to Spec(\mathcal{Z})$ is the projection, and we do not distinguish between a module over a commutative algebra and the corresponding quasicoherent sheaf over its spectrum.

It is easy to lift this quasiisomorphism to a quasiisomorphism in the category of G-bimodules. More precisely, if N lies in the heart of the derived category

(i.e. $H^i(N) = 0$ for $i \neq 0$; the only case we need $N = \mathcal{H}$ satisfies this assumption) then $RHom_{\mathcal{Z}\otimes\mathcal{Z}}(\mathcal{Z}, N \otimes D)$ has only non-trivial cohomology in degree 0 isomorphic to N, hence in this case we have a canonical quasiisomorphism $N \leftarrow \tau_{\leqslant 0}(RHom_{\mathcal{Z}\otimes\mathcal{Z}}(\mathcal{Z}, N \otimes D)) \to RHom_{\mathcal{Z}\otimes\mathcal{Z}}(\mathcal{Z}, N \otimes D)$. If $N^{\cdot}$ is any complex (of injective modules) and $D^{\cdot}$ is any complex of injective $\mathcal{Z}$-modules quasiisomorphic to the dualizing sheaf, then $RHom_{\mathcal{Z}\otimes\mathcal{Z}}(\mathcal{Z}, N \otimes D)$ is represented by the bicomplex $Hom_{\mathcal{Z}\otimes\mathcal{Z}}(\mathcal{Z}, N^{\cdot} \otimes D^{\cdot})$; we can apply the above canonical quasiisomorphism in each column of this bicomplex to get the desired quasiisomorphism for any N. $\square$

4.18. *Proof* of Theorem 4.13. Since $I(M) \in \mathcal{R}_{\mathcal{Z}}$ for $M \in \mathcal{M}$ we see using 4.17 that

$$D_h(M) = RHom_G(M, \mathcal{H}) \cong RHom_{G\times\mathcal{Z}}(I(M), \mathcal{H} \otimes D)$$

(more precisely we apply the forgetful functor $G \times \mathcal{Z} - mod \to G - mod$ to the isomorphism 4.17 to get the last isomorphism). By 4.15 we have:

$$RHom_{G\times\mathcal{Z}}(I(M), \mathcal{H} \otimes D) \cong DL(RHom_{\mathcal{Z}}(M, D))$$

The Theorem is proved. $\square$

2. Elliptic pairing and filtrations on the Hecke algebra

Preliminaries and notations. In this chapter we prove

Theorem 0.19. *The equality* (2)*:*

$$O_{g^{-1}}(\langle\rho\rangle) = \chi_\rho(g)$$

holds for any $\rho \in \mathcal{R}$*, and any elliptic element* $g \in G$*.*

and deduce from it

Theorem 0.20. *Assume* $\mathrm{char}(F) = 0$ *and* $k = \mathbb{C}$*. The equality* (1)*:*

$$\sum(-1)^i \dim Ext^i(\rho_1, \rho_2) = \int_{Ell} \chi_{\rho_1}(g^{-1})\chi_{\rho_2}(g)d\mu(g)$$

is valid for all ρ_1*,* $\rho_2 \in \mathcal{R}$*.*

It will be convenient to allow G to be an arbitrary reductive group (not requiring that G has compact center). Notice however that a group with non-compact center does not contain elliptic elements (in the terminology we adopt), so the RHS of (1) vanishes; it is well-known (see Claim 4.3 below) that so does the LHS. The statement of Theorem 0.19 is vacuous for a group with noncompact center.

0.21. *Hattori-Stallings trace for the Hecke algebra.* Since the Hecke algebra $\mathcal{H}$ is not unital, a comment on the definition of the Hattori-Stallings trace appearing in the formulation of Theorem 0.19 is required.

Recall from [BD] that for a *nice* open compact subgroup $K \subset G$ (see [BD] 2.1b) for the definition) the full subcategory $\mathcal{M}^K \subset \mathcal{M}$ consisting of modules generated by K-fixed vectors is a direct summand in $\mathcal{M}$. Moreover, the functors $\mathfrak{M} \mapsto \mathfrak{M}^K$ and $M \to M \otimes_{\mathcal{H}(G,K)} \mathcal{H}$ are mutually inverse equivalences between $\mathcal{M}^K$ and the category of $\mathcal{H}(G,K)$-modules (here $\mathcal{H}(G,K)$ is the subalgebra of K-biinvariant measures in $\mathcal{H}$).

Hence $\mathcal{H}(G,K)$ is a Noetherian unital algebra of finite homological dimension, so the usual construction of the Hattori-Stallings trace applies to it. Now for any $\mathfrak{M} \in \mathcal{M}^K$, $E \in End_G(\mathfrak{M})$ and $K' \subset K$ we have $\mathfrak{M}^{K'} = \mathfrak{M}^K \otimes_{\mathcal{H}(G,K)} \mathcal{H}(G,K')$, hence the natural map $\mathcal{H}(G,K)/[,] \to \mathcal{H}(G,K')/[,]$ induced by the imbedding $\mathcal{H}(G,K) \hookrightarrow \mathcal{H}(G,K')$ sends $Tr_{H-St}(\mathfrak{M}^K, E|_{\mathfrak{M}^K})$ to $Tr_{H-St}(\mathfrak{M}^{K'}, E|_{\mathfrak{M}^{K'}})$. Thus for small enough K' the image of $Tr_{H-St}(\mathfrak{M}^{K'}, E|_{\mathfrak{M}^{K'}})$ in $\mathcal{H}/[\mathcal{H},\mathcal{H}]$ does not depend on K'; this is by the definition the Hattori-Stallings trace of $(\mathfrak{M}, E)$.

0.22. A regular elliptic element $g \in G$ is fixed throughout the argument.

Since g is elliptic it is also compact; we fix a maximal compact subgroup K_0 containing g.

Let $K \subset K_0$ be a normal subgroup; since g normalizes K it acts on the set of double cosets $K\backslash G/H$ for any subgroup $H \subset G$. We say that g is *K-elliptic* if it acts on $K\backslash G/P$ without fixed points for any proper parabolic $P \subset G$. Obviously g is K-elliptic for small enough normal open $K \subset K_0$.

We say that K is *nice* if it is nice with respect to some minimal Levi subgroup L_0, which is in good position with K_0 (see 0.24, 2.1 below).

We fix a normal open subgroup $K \subset K_0$, such that g is K-elliptic.

From now on we will denote by $\mathcal{H}$ *the subalgebra* $\mathcal{H}(G,K) \subset \mathcal{H}(G)$ *of* K*-biinvariant distributions.*

We will in fact work with the following form of Theorem 0.19

Theorem 0.23. *One can define for any $g \in Ell$, and any finitely generated $\mathcal{H}$-module M a number $Tr(g, M) \in k$ so that*

i) $Tr(g, M)$ is additive on short exact sequences in M.

Fix $\mathfrak{M} \in \mathcal{M}$.

ii) If $\mathfrak{M}$ is admissible, and its character $\chi_{\mathfrak{M}}$ is constant on the coset $K \cdot g$, then $Tr(g, \mathfrak{M}^K) = \chi_{\mathfrak{M}}(g)$ is the character value.

iii) Let M be a projective $\mathcal{H}$-module. Then there exists an idempotent $E \in Mat_m(\mathcal{H})$ such that M is isomorphic to the image of E acting on $\mathcal{H}^{\oplus m}$ on the right.

Suppose that K is nice, and the locally constant invariant function $x \to O_{x^{-1}}(\langle \mathfrak{M} \rangle)$ defined on the regular elliptic set is (defined and) constant on the coset $K \cdot g$. Then we have

$$Tr(g, M) = O_{g^{-1}}(\sum E_{ii}).$$

It is immediate to deduce 0.19 from 0.23 (see 3.8).

The next three sections are devoted to the construction of $Tr(g, M)$ and verification of the properties i)–iii).

0.24. *Notations.* Let us choose a minimal parabolic with Levi decomposition $P_0 = L_0 \cdot U_0 \subset G$ so that L_0 is in good relative position with K_0, i.e. the point p of the Bruhat-Tits building fixed by K_0 lies in the apartment normalized by L_0.

A standard parabolic is the one containing P_0; a standard Levi is the Levi subgroup of a standard parabolic containing L_0.

The letters P, L, U with indices will be reserved for a standard parabolic, its standard Levi, and its unipotent radical respectively, unless stated otherwise.

Let $W = Nm(L_0)/L_0$, $W_L = (Nm(L_0) \cap L)/L_0$ and $W_{aff} = Nm(L_0)/L_0^c$ be respectively the Weyl group, the Weyl group of L and the affine Weyl group.

For an L-module ρ we will sometimes write $i_L^G(\rho)$ instead of $i_P^G(\rho)$ for the parabolically induced module (of course there is no ambiguity here, because P and L determine each other uniquely).

Let A_L be the center of L, X_L be the lattice of coweights of A_L. Thus $X := X_{L_0}$ is the lattice of abstract coweights of G. Set $\mathfrak{a} = X \otimes \mathbb{R}$; let $\mathfrak{a}_+ \subset \mathfrak{a}$ (respectively $\mathfrak{a}^+$) be the dominant Weyl chamber (respectively its closure). For a subset $? \subset \mathfrak{a}$ we denote the intersection $? \cap \mathfrak{a}^+$ by $?^+$.

For a standard Levi L let $\mathfrak{a}_L \subset \mathfrak{a}$ denote $X_L \otimes \mathbb{R}$.

Fixing a uniformizer $\mathfrak{p} \in F^\times$ we obtain a canonical imbedding $\iota_L : X_L \hookrightarrow A_L$, $\chi \mapsto \chi(\mathfrak{p})$, inducing an imbedding of finite index $X_L \hookrightarrow L/L^c$.

For a topological group H we denote by H^c the subgroup generated by all compact subgroups.

Consider the composition $L/L^c \hookrightarrow (L/L^c) \otimes \mathbb{R} = \mathfrak{a}_L \hookrightarrow \mathfrak{a}$. We identify L/L^c with its image under this map.

By $\mathfrak{P}_L$ we denote the orthogonal projection $\mathfrak{a} \to \mathfrak{a}_L$.

By a root we will mean a nonmultipliable root, and by a coroot the corresponding coweight of A_{L_0}. Thus coroots (respectively, roots) form a reduced root system in $\mathfrak{a}$ (respectively $\mathfrak{a}^*$) (see [BoT], Theorem 7.2). The choice of a minimal parabolic P_0 determines the basis of simple (co)roots, and the dual basis of fundamental (co)weights.

Let Σ^+ be the set of positive coroots; for a Levi L let $\Sigma^+_L \subset \Sigma^+$ be the set of positive coroots of the derived group $L^{[,]}$.

If α is a simple coroot, ω_α is the dual fundamental weight, and λ_1, $\lambda_2 \in X$ then we will write $\lambda_1 \underset{\alpha}{\leqslant} \lambda_2$ meaning $(\lambda_1, \omega_\alpha) \leqslant (\lambda_2, \omega_\alpha)$. We also write $\lambda \underset{\alpha}{\leqslant} n$ meaning $(\lambda, \omega_\alpha) \leqslant n$.

The standard partial order $\preccurlyeq$ on $\mathfrak{a}$ is defined by $\lambda_1 \preccurlyeq \lambda_2$ if $\lambda_1 \underset{\alpha}{\leqslant} \lambda_2$ for all α; as usual we write $\lambda_1 \prec \lambda_2$ instead of $\lambda_1 \preccurlyeq \lambda_2$ & $\lambda_1 \neq \lambda_2$ etc.

Recall the notation $k[S] := \bigoplus\limits_{s \in S} k$. When S is a (semi)group this is the (semi)group algebra; for $s \in S$ let $[s] \in k[S]$ be the delta-function of s.

If $(\Lambda, \blacktriangleleft)$ is a (partially) ordered semigroup, we use the notation $\Lambda_{\blacktriangleleft \lambda} := \{\mu \in X \mid \mu \blacktriangleleft \lambda\}$ for $\lambda \in \Lambda$.

We have a $(\Lambda, \blacktriangleleft)$-filtration on the semigroup algebra $k[\Lambda]$, defined by $k[\Lambda]_{\blacktriangleleft \lambda} := k[\Lambda_{\blacktriangleleft \lambda}]$.

Analogous notations apply when $\blacktriangleleft$ is a filtration on Λ indexed by another (partially) ordered semigroup.

We use the same symbol for a partial order on $\mathfrak{a}$ and its restriction to a sublattice.

We will say that $\lambda \in X_L$ is dominant (or large) enough if $(\lambda, r) \gg 0$ when r is a root of U. (The latter condition will be abbreviated as $\langle \lambda, \Sigma^+ - \Sigma^+_L \rangle \gg 0$.

1. **"Spectral" filtration: definition of the trace functional.**

1.1. Let ρ be an admissible representation of L. We can form a (non-admissible) G-module $\Pi_\rho := i_L^G(\rho \otimes k[L/L^c]) = ind^G_{L^c \cdot U}(\rho|_{L^c})$. Then Π_ρ has also a $k[L/L^c]$-action commuting with the G-action, which comes from the action of $k[L/L^c]$ on the second multiple of $\rho \otimes k[L/L^c]$:

$$l(f)(g) = (Id \otimes l)(f(g)) \tag{17}$$

for $l \in L/L^c$, $f : G \to \rho \otimes k[L/L^c]$, $f \in \Pi_\rho$. For $\lambda \in L/L^c$ let $[\lambda]_\rho \in \mathrm{End}(\Pi_\rho)$ denote action (17).

For an unramified character $\psi : L/L^c \to k^\times$ we have a canonical isomorphism of G-modules (but not of L/L^c-modules) $I_\psi : \Pi_\rho \widetilde{\longrightarrow} \Pi_{\rho \otimes \psi}$, satisfying

$$I_\psi(l(m)) = \psi(l) l(I_\psi(m)) \tag{18}$$

for $l \in L/L^c$, $m \in \Pi_\rho$.

From (18) it follows that $E \in \mathrm{End}(\Pi_\rho)$ commutes with λ_ρ for all $\lambda \in L/L^c$ iff $I_\psi \circ E \circ I_\psi^{-1}$ commutes with all $[\lambda]_{\rho \otimes \psi}$, thus we have a canonical isomorphism

$$\mathrm{End}_{L/L^c}(\Pi_\rho) \cong \mathrm{End}_{L/L^c}(\Pi_{\rho \otimes \psi}). \tag{19}$$

For any standard Levi L let us choose a set $Cusp_L$ of cuspidal irreducible representations of L, in such a way that any cuspidal irreducible representation of L is isomorphic to a unique representation $\rho \otimes \psi$, where $\rho \in Cusp_L$, and $\psi : L/L^c \to k$ is an unramified character of L. We can (and will) make this choice in such a way that $\rho|_{\iota_L(X_L)}$ is trivial for $\rho \in Cusp_L$.

Let $Cusp^K_L \subset Cusp_L$ be the set of such $\rho \in Cusp_L$ that $i^G_L(\rho)^K \neq 0$; set $Cusp := \bigcup\limits_L Cusp_L$ and $Cusp^K := \bigcup\limits_L Cusp^K_L$.

Set $\tilde{\mathcal{H}} := \bigoplus\limits_{\rho \in Cusp^K} \mathrm{End}_{k[L/L^c]}(\Pi^K_\rho)$.

We have $[\lambda]_\rho \in \tilde{\mathcal{H}}$ for $\rho \in Cusp_L^K$. For $\lambda \in X$ define $[\lambda] \in \tilde{\mathcal{H}}$ by

$$\lambda = \sum_L \sum_{\rho \in Cusp_L^K} [\lambda]_\rho, \tag{20}$$

where L runs over the set of standard Levi subgroups, such that $\lambda \in X_L$.

We have an imbedding $I = \sum_{\rho \in Cusp^K} I_\rho : \mathcal{H} \hookrightarrow \tilde{\mathcal{H}}$, where $I_\rho : \mathcal{H} \to \mathrm{End}(\Pi_\rho^K)$.

1.2. We are going to define an $\mathfrak{a}$-multifiltration on $\tilde{\mathcal{H}}$. To clarify the idea we first carry out the construction under an additional simplifying assumption that the maximal compact subgroup K_0 containing g is good, i.e. $G = K_0 \cdot P_0$ (the general case will be treated in the next subsection).

If K_0 is good, then the $k[L/L^c]$-module Π_ρ can be canonically trivialized in the following fashion.

By the definition the space of the induced representation Π_ρ is the space of functions on G with values in $\rho \otimes k[L/L^c]$ which transform accordingly under the right action of P. Let us say that $f \in \Pi_\rho$ is constant if $f|_{K_0}$ takes values in $\rho \otimes 1 \subset \rho \otimes k[L/L^c]$. Then the space of constant elements is a K_0-submodule, and generates Π_ρ freely as a $k[L/L^c]$-module. We denote the space of constant elements in Π_ρ by $(\Pi_\rho)_0$. For an unramified character ψ of L the isomorphism I_ψ sends $(\Pi_\rho)_0$ to $(\Pi_{\rho\otimes\psi})_0$, and we have the canonical isomorphism $(\Pi_{P,\rho})_0 \widetilde{\longrightarrow} i_P^G(\psi \otimes \rho) = \Pi_\rho \otimes_{k[L/L^c]} \psi$.

We abbreviate $(\Pi_\rho)_0^K = i_0(\rho)$. So we have $\tilde{\mathcal{H}} = \bigoplus_{Cusp} \mathrm{End}_{k[L/L^c]}(\Pi_\rho^K) = \bigoplus_{Cusp^K} \mathrm{End}(i_0(\rho)) \otimes k[L/L^c]$.

For a (partial) order $\blacktriangleleft$ on $\mathfrak{a}$ we have an $(\mathfrak{a}, \blacktriangleleft)$-filtration on $\tilde{\mathcal{H}}$ given by $\tilde{\mathcal{H}}_{\blacktriangleleft\lambda} := \bigoplus_{\rho \in Cusp^K} \mathrm{End}(i_0(\rho)) \otimes k[L/L^c]_{\blacktriangleleft\lambda}$.

1.3. Consider now the case when K_0 is an arbitrary maximal compact subgroup.

Recall that p is the point of the Bruhat-Tits building X fixed by K_0; let $\mathfrak{A} \ni p$ be the apartment corresponding to L_0. Let $W_p = K_0 \cap Nm(L_0)/L_0^c \subset W_{aff}$ be the stabilizer of p, and $W_{aff}^L \subset W_{aff}$ be the subgroup $(L^c \cap Nm(L_0))/L_0^c$. The groups $Nm(L_0)$, $L^c \cap Nm(L_0)$ act on $\mathfrak{A}$ respectively through W_{aff}, W_{aff}^L.

Notice that $P^c = L^c \cdot U$.

Lemma 1.4. *The map $w \mapsto K_0 w P^c$ provides a bijection*

$$W_p \backslash W_{aff}/W_{aff}^L \widetilde{\longrightarrow} K_0 \backslash G/P^c \ . \tag{21}$$

Proof First notice that since K_0, L_0 are in good relative position we have $K_0 \supset L_0^c$, and obviously $L_0^c \subset P^c$, so the expression $K_0 w P^c$ is meaningful.

Let $W_{aff}(L)$ denote $Nm_L(L_0)/L_0^c$. On the LHS of (21) the group $W_{aff}(L)/W_{aff}^L$ acts from the right. From the Bruhat decomposition $L = (P_0 \cap L) \cdot Nm_L(L_0) \cdot (P_0 \cap L)$ we see, using the inclusion $P_0 \cap L = L_0 \cdot (U \cap L) \subset L_0 \cdot L^c$, that the map $W_{aff}(L)/W_{aff}^L = Nm_L(L_0)/Nm_{L^c}(L_0) \to L/L^c$ is an isomorphism.

On the RHS of (21) the group $L/L^c = P/P^c$ acts from the right. The map (21) obviously agrees with the action of L/L^c. Moreover, we claim that the action on both sides is free. Indeed, the right action of $L/L^c = P/P^c$ on G/P^c is obviously

free; since the map $G/P^c \to K_0\backslash G/P^c$ is proper, the stabilizer of any point in $K_0\backslash G/P^c$ is compact. However L/L^c is a (discrete) free abelian group, so it does not contain compact subgroups. The same argument applies to the LHS (with "finite-to-one" replacing "proper").

Thus it is enough to prove that the map $W_p\backslash W_{aff}/W_{aff}(L) \to K_0\backslash G/P$ is an isomorphism. This can be reformulated via the Bruhat-Tits building X as follows.

Any points $x \in X$, $z \in \overline{X} - X$ lie in the closure of some apartment $\mathfrak{A}$. Moreover, if $x' \in \mathfrak{A}$, $z' \in \overline{\mathfrak{A}} - \mathfrak{A}$ is another pair of points, such that $x' = g(x)$, $z' = g(z)$, then also $x' = w(x)$, $z' = w(z)$ for some element $w \in W_{\mathfrak{A}}$, where $W_{\mathfrak{A}} = Nm(\mathfrak{A})/Stab(\mathfrak{A}) \cong W_{aff}$.

The fact that x and z lie in one apartment follows from [BT1] 7.4.18(ii). To check the second statement recall that by Lemma 3.8 of part 1 there exists a unique geodesic ray connecting a point in X and a point in $\overline{X} - X$. In particular the rays $[x, z)$, $g([x, z)) = [x', z')$ lie in $\mathfrak{A}$ since there endpoints lie in $\overline{\mathfrak{A}}$. Hence by [BT1] 7.4.8 there exists $w \in W_{\mathfrak{A}}$ such that $g|_{[x,z)} = w|_{[x,z)}$. By continuity of the action of g, w we get $g(z) = w(z)$. □

1.5. For $x \in W_p\backslash W_{aff}/W_{aff}^L$ let $(G/U)_x$ be the corresponding orbit of $K_0 \times P^c$.

Let the subgroup $W_f^p \subset \mathrm{Aut}(\mathfrak{A})$ be generated by reflections in all hyperplanes parallel to root hyperplanes and passing through p; let W_{aff}^p be generated by W_{aff} and W_f^p. Let Λ be the intersection of W_{aff}^p with the group of translations; it is obvious that Λ is a lattice in $\mathfrak{a} = X \otimes \mathbb{R}$ containing X, and $W_{aff}^p = W_f^p \ltimes \Lambda$.

Consider the projection $\pi_L : W_p\backslash W_{aff}/W_{aff}^L \to W_f^p\backslash W_{aff}^p/W_{aff}^L = \Lambda/W_{aff}^L \xrightarrow{\mathfrak{P}_L} \mathfrak{a}_L \subset \mathfrak{a}$; denote its image by Λ_L. For $\lambda \in \Lambda_L$ set $(G/U)_\lambda = \bigcup\limits_{\pi_L(x)=\lambda} (G/U)_x$.

Now we get a Λ_L-grading on Π_ρ for $\rho \in Cusp_L$ as follows

$$\Pi_\rho = C_c(G/U)^K \otimes_{\mathcal{H}(L^c)} \rho = \bigoplus_{\lambda \in \Lambda_L} C_c(G/U)_\lambda^K \otimes_{\mathcal{H}(L^c)} \rho. \tag{22}$$

For $\lambda \in \mathfrak{a}$ we now set

$$\tilde{\mathcal{H}}_\lambda = \{h \in \tilde{\mathcal{H}} \,|\, h(\Pi_\rho^K)_\nu \subset (\Pi_\rho^K)_{\nu+\lambda}\}, \tag{23}$$

$$\tilde{\mathcal{H}}_{\blacktriangleleft\lambda} = \bigoplus_{\mu \blacktriangleleft \lambda} \tilde{\mathcal{H}}_\mu. \tag{24}$$

Comparison of the definitions shows that the latter filtration coincides with the one introduced in 1.1 when K_0 is good.

1.6. For a (partial) order $\blacktriangleleft$ on $\mathfrak{a}$ we have defined an $(\mathfrak{a}, \blacktriangleleft)$-filtration on $\tilde{\mathcal{H}}$. Let $\mathcal{H}_{\blacktriangleleft\lambda}$ be the induced filtration on $\mathcal{H} \subset \tilde{\mathcal{H}}$.

The next Proposition is the key technical result needed for our definition of the trace functional.

For a standard Levi L_1, $\rho \in Cusp_{L_1}$ the space of Π_ρ is the space of $\rho \otimes k[L_1/L_1^c]$-valued functions on G transforming accordingly under the right action of P_1; for $L \supset L_1$ and $x \in K\backslash G/P$ let $m_x^\rho \in \mathrm{End}(\Pi_\rho^K)$ be multiplication by the δ-function of the corresponding right $K \times P$ double coset.

Set also $m_x = \sum\limits_{L' \subset L} \sum\limits_{\rho \in Cusp_{L'}^K} m_x^\rho$.

Proposition 1.7. *Fix a standard Levi $L \supset L_1$, $\rho \in Cusp^K_{L_1}$, $x \in K\backslash G/P$ and $a \in \mathbb{R}_{>0}$. Then for dominant enough $\lambda \in X^+_L$ there exists an element $h^\rho_{x,\lambda} \in \mathcal{H}$ such that $I(h^\rho_{x,\lambda}) = \sum\limits_{L_2=w(L_1)\subset L} [\lambda]_{w(\rho)} \cdot m_x^{w(\rho)} + h'$, where $h' \in \sum\limits_{r\in\Sigma^+-\Sigma^+_L} \tilde{\mathcal{H}}_{\preccurlyeq\lambda-a\cdot r}$.*

Remark 1.8. The construction of the trace functional will use only the fact $I(h^\rho_{x,\lambda}) - \sum\limits_{L_2=w(L_1)\subset L} [\lambda]_{w(\rho)} \cdot m_x^{w(\rho)} \in \tilde{\mathcal{H}}_{\prec\lambda}$. The stronger condition stated in the Proposition will play a role only in the proof that the constructed functional satisfies property iii) of 0.23.

We will deduce the Proposition from the *matrix Payley-Wiener Theorem* which describes the image of the imbedding I (see [BR], §21). Let us first recall its contents.

If M_1, M_2 are two modules over a free abelian group $\mathbb{Z}^n$ then by a rational morphism from M_1 to M_2 we will mean an element of $Hom_{Frac(k[\mathbb{Z}^n])}\big(M_1 \otimes_{k[\mathbb{Z}^n]} Frac(k[\mathbb{Z}^n]), M_2 \otimes_{k[\mathbb{Z}^n]} Frac(k[\mathbb{Z}^n])\big)$ where $Frac(?)$ is the field of fractions of a commutative ring ?.

We recall the necessary part of the theory of intertwining operators. Suppose that standard Levi L_1, L_2 are conjugate, so we have $L_2 = w(L_1)$ for some $w \in W$.

Then there exists a canonical rational isomorphism of G-modules $I_w = I^\rho_w : \Pi_\rho \to \Pi_{w(\rho)}$, such that $I_w \circ [\lambda]_\rho = [w(\lambda)]_{w(\rho)} \circ I_w$ for $\lambda \in L/L^c$ satisfying:

i) Transitivity: if $L_2 = w_1(L_1)$, $L_3 = w_2(L_2)$ are standard Levi then $I_{w_2w_1}(m) = I_{w_2} \circ I_{w_1}(f_{w_1,w_2} m)$ for some rational function $f_{w_1,w_2} \in Frac(k[L_1/L_1^c])$.

ii) Compatibility with induction: if $L \supset L_1, L_2$, $L_2 = w(L_1)$ then for a representation $\rho \in Cusp_{L_1}$ we have $I^\rho_w = i^G_L(I^\rho_{w,L})$, where $I^\rho_{w,L} : i^L_{L_1}(\rho \otimes k[L_1/L_1^c]) \to i^L_{L_2}(\rho \otimes k[L_2/L_2^c])$ is the intertwining operator between the L-modules.

We define the rational morphisms $R^\rho_w : \mathrm{End}_{L_1/L_1^c}(\Pi_\rho) \to \mathrm{End}_{L_2/L_2^c}(\Pi_{w(\rho)})$ by $R_w(E) = I_w \circ E \circ I_w^{-1}$. Notice that the operators R_w satisfy the stronger transitivity condition:

$$R_{w_1w_2} = R_{w_1} \circ R_{w_2}.$$

Let L_1, L_2 be standard Levi subgroups such that $L_2 = w(L_1)$ and $\rho_1 \in Cusp_{L_1}$, $\rho_2 \in Cusp_{L_2}$ be such that $\rho_2 = w(\rho_1) \otimes \psi$ for an unramified character ψ. Identify $\mathrm{End}_{L_2/L_2^c}(\Pi_{w(\rho_1)})$ with $\mathrm{End}_{L_2/L_2^c}(\Pi_{\rho_2})$ by means of (19); then R^ρ_w provides a rational morphism $\mathrm{End}_{L_1/L_1^c}(\Pi_{\rho_1}) \to \mathrm{End}_{L_2/L_2^c}(\Pi_{\rho_2})$. We denote it again by R_w or R^ρ_w, this abuse of notation hopefully will not lead to a confusion.

For future reference we record the following

Lemma 1.9. *For standard Levi subgroups $L_2 = w(L_1)$ and $\rho_1 \in Cusp_{L_1}$, $\rho_2 \in Cusp_{L_2}$ such that $\rho_2 = w(\rho_1) \otimes \psi$ as above we have $R_w([\lambda]_{\rho_1}) = [w(\lambda)]_{\rho_2}$ provided λ lies in $X_{L_1} \subset L_1/L_1^c$.*

Proof Since $R^{\rho_1}_w([\lambda]_{\rho_1}) = [w(\lambda)]_{w(\rho_1)}$ we have only to check that $[w(\lambda)]_{w(\rho_1)} = [w(\lambda)]_{\rho_2}$ (more precisely, that the two sides agree under (19)). By (18) this reduces to $\psi(\mu) = 1$ for $\mu \in X_{L_2}$. But this is clear because X_{L_2} acts trivially both on $w(\rho_1)$ and on $\rho_2 \in Cusp_{L_2}$. □

There exists also another type of operators acting on $\bigoplus\limits_{Cusp} \Pi_\rho$. Namely, for a given irreducible representation ρ of L there might exist a finite number of unramified characters ψ such that $\psi \otimes \rho \cong \rho$. For such ψ the canonical isomorphism

$I_\psi : \Pi_\rho \widetilde{\longrightarrow} \Pi_{\rho\otimes\psi}$ induces an automorphism of Π_ρ. This automorphism is defined uniquely up to a constant (we have to choose an isomorphism $J : \rho\otimes\psi \widetilde{\longrightarrow} \rho$, and then get $i_L^G(J)\circ I_\psi : \Pi_\rho \widetilde{\longrightarrow} \Pi_\rho$); thus we have a uniquely defined automorphism T_ψ of $\mathrm{End}_{L/L^c}(\Pi_\rho)$, $T_\psi : E \mapsto (i_L^G(J)\circ I_\psi)\circ E \circ (i_L^G(J)\circ I_\psi)^{-1}$.

The matrix Payley-Wiener Theorem ([BR], Theorem 33) asserts that

An element $E = \sum\limits_{Cusp^K} E_\rho \in \tilde{\mathcal{H}}$ *lies in the image of* I *iff*

i) *For any conjugate standard Levi subgroups* $L_2 = w(L_1)$ *and* $\rho_1 \in Cusp_{L_1}$, $\rho_2 \in Cusp_{L_2}$ *such that* $\rho_2 \cong w(\rho_1\otimes\psi)$ *for an unramified character* ψ *we have* $R_w^{\rho_1}(E_{\rho_1}) = E_{\rho_2}$.

ii) $T_\psi(E_\rho) = E_\rho$ *whenever* $\psi\otimes\rho\cong\rho$.

1.10. We return to the proof of 1.7.

For standard Levi subgroups $L' = w(L)$ and $\rho\in Cusp_L^K$ we will not distinguish between $\Pi_w(\rho)$ and $\Pi_{w(\rho)\otimes\psi_w(\rho)}$ when this is not likely to lead to a confusion. We will also abuse notations writing R_w instead of $T_{\psi_{w(\rho)}}\circ R_w$. Thus $R_w : \mathrm{End}(\Pi_\rho)\to \mathrm{End}(\Pi_{w(\rho)})$ is a rational endomorphism of $\tilde{\mathcal{H}}$.

The plan is to start with $m_x^\rho\cdot[\lambda]_\rho$ and then average over the set of intertwining operators; we have to take care of the poles of intertwining operators, and also to ensure that we really get an element with the desired highest term. It will be not hard to see that each summand in the resulting sum is T_ψ-invariant.

Lemma 1.11. *If* $L_2 = w(L_1)$, $L\supset L_1, L_2$ *are standard Levi,* $x\in K\backslash G/P$, $\lambda\in X_L\subset X_{L_1}, X_{L_2}$ *then we have* $R_w(m_x^\rho\cdot[\lambda]_\rho) = m_x^\rho\cdot[\lambda]_\rho$ *for any* ρ.

Proof By 1.9 the action by intertwining operators induces the standard geometric action on coweights, thus $R_w([\lambda]_\rho) = [\lambda]_{w(\rho)}$ for $\lambda\in X_L$ and $w\in W_L$.

If we think of $\Pi_\rho = i_{L_1}^G(\rho\otimes k[L_1/L_1^c]) = i_L^G\left(i_{L_1}^L(\rho\otimes k[L_1/L_1^c])\right)$ as sections of the corresponding sheaf on G/P then the action of m_x^ρ comes from an action on the sheaf; moreover the corresponding endomorphism of a stalk is either identity or 0. From $R_w(Id) = Id$ we conclude by compatibility that it sends m_x^ρ to $m_x^{w(\rho)}$. ◻

For $h\in\tilde{\mathcal{H}}$, $h = \sum\limits_{L,\lambda\in\Lambda_L} h_\lambda$, where $h_\lambda\in\tilde{\mathcal{H}}_\lambda$, let the support of h be

$$\mathrm{supp}(h) := \{\lambda\in\mathfrak{a} \mid h_\lambda\neq 0\}. \tag{25}$$

1.12. Consider the following situation. Let $L_1\neq L_2$ be two conjugate standard Levi lying in a standard Levi L with $\mathrm{rank}(L) = \mathrm{rank}(L_1)+1 = \mathrm{rank}(L_2)+1$. Then necessary $L_2 = w_0^L(L_1)$, where $w_0^L\in W_L$ is the longest element. Let α be a nonzero element of the group $X_{L_1}\cap(\mathfrak{a}_L)^\perp$. Notice that α is proportional to $\mathfrak{P}_{L_1}(r)$ where r is the only simple coroot of L which does not lie in Σ_{L_1}.

Lemma 1.13. *a) Components of the divisor of poles of* $R_{w_0^L}$ *have the form* $[\alpha]_\rho = \mathrm{const}$; *in other words for some* $f\in k[t]$ *for any* $h\in\mathrm{End}_{L_2/L_2^c}\Pi^K_{w_0^L(\rho)}$ *we have* $R_{w_0^L}(f([\alpha]_\rho)h)\in\mathrm{End}_{L_2/L_2^c}\Pi^K_{w_0^L(\rho)}\subset\mathrm{End}_{L_2/L_2^c}\Pi^K_{w_0^L(\rho)}\otimes Frac(k[L_2/L_2^c])$.

b) There exists $n_0\in\mathbb{Z}$ *such that the following holds. Suppose that* $h\in\mathrm{End}_{L_1/L_1^c}\Pi_\rho^K$ *is such that* $R_{w_0^L}(h)$ *is regular, i.e. belongs to* $\mathrm{End}_{L_2/L_2^c}\Pi^K_{w_0^L(\rho)}$ *rather than to* $\mathrm{End}_{L_2/L_2^c}\Pi^K_{w_0^L(\rho)}\otimes Frac(k[L_2/L_2^c])$. *Then* $\nu\in\mathrm{supp}(R_{w_0^L}(h))\Rightarrow(w_0^L)^{-1}(\nu)+n\alpha\in\mathrm{supp}(h)$ *for some* n *with* $|n|\leqslant n_0$.

Proof a) We can decompose L as an almost direct product of its center A_L and derived group $L^{[,]}$, i.e. we have a homomorphism $\pi : A_L \times L^{[,]} \to L$ with finite kernel and cokernel. Obviously for $j = 1, 2$ the pull-back of the representation $i^L_{L_j}(k[L_j/L^c_j] \otimes \rho)$ under π decomposes as the tensor product of the free module $k[A_L/A^c_L]$ with the parabolically induced module $i^{L^{[,]}}_{L_j \cap L^{[,]}}(\bigoplus_{L_j/L^c_j \cdot A_L \cdot (L^{[,]} \cap L_j)} k[(L_j \cap L^{[,]})/(L_j \cap L^{[,]})^c] \otimes \rho|_{L_j \cap L^{[,]}})$. The intertwining operator $I^\rho_{w^L_0, L}$ also decomposes as $Id \otimes I^{\rho|_{L_1 \cap L^{[,]}}}_{w^L_0, L^{[,]}}$, hence so does $R^\rho_{w^L_0, L}$. Thus the image of the divisor of poles of $R^\rho_{w^L_0, L}$ under the finite covering $Spec(k[L/L^c]) \to Spec(k[A_L/A^c_L]) \times Spec(k[(L^{[,]} \cap L_1)/(L^{[,]} \cap L_1)^c])$ is the product of $Spec(k[A_L/A^c_L])$ with a finite subscheme of $Spec(k[(L^{[,]} \cap L_1)/(L^{[,]} \cap L_1)^c])$ (notice that $(L^{[,]} \cap L_1)/(L^{[,]} \cap L_1)^c$ is a free cyclic group, so $Spec(k[(L^{[,]} \cap L_1)/(L^{[,]} \cap L_1)^c])$ is the punctured affine line). This finite subscheme is necessary killed by the function $\prod_{j=1}^{n}([\alpha]_\rho - q_j)$ for some finite set of elements $q_1, ..., q_n \in k^\times$, hence the same holds for the divisor of poles of $R^\rho_{w^L_0, L}$. a) follows by compatibility of the intertwining operators with induction.

Let us prove b). For $j = 1, 2$ consider the projection $pr_j : K\backslash G/P^c_j \to K\backslash G/P^c$. For $y \in K\backslash G/P^c$ denote $(\Pi^K_\rho)_y := \bigoplus_{\pi_1(x)=y} (\Pi^K_\rho)_x$, $(\Pi^K_{w^L_0(\rho)})_y := \bigoplus_{\pi_2(x)=y} (\Pi^K_{w^L_0(\rho)})_x$. Notice that $(\Pi^K_\rho)_y$ is a $k[\mathbb{Z}\alpha]$-submodule for any $y \in K\backslash G/P^c$.

By a) the intertwining operator $I_{w^L_0}$ is a well-defined morphism $\Pi^K_\rho \otimes_{k[\mathbb{Z}\alpha]} k([\alpha]_\rho) \to \Pi^K_{w^L_0(\rho)} \otimes_{k[\mathbb{Z}w^L_0(\alpha)]} k([w^L_0(\alpha)]_{w^L_0(\rho)})$. From compatibility with induction it follows that $I_{w^L_0}(\Pi^K_\rho)_y \otimes k([\alpha]_\rho) \subset (\Pi^K_{w^L_0(\rho)})_y \otimes k([w^L_0(\alpha)]_{w^L_0(\rho)})$.

Now from the definition it is easy to see that for $j = 1, 2$ the following diagram is commutative:

$$\begin{array}{ccc} K\backslash G/P^c_j & \xrightarrow{pr_j} & K\backslash G/P^c \\ {\scriptstyle \pi_{L_j}}\downarrow & & {\scriptstyle \pi_L}\downarrow \\ \Lambda_{L_j} & \xrightarrow{\mathfrak{P}_{L_j}} & \Lambda_L \end{array}$$

Hence for $\mu \in \Lambda_L$ we have $k[\mathbb{Z}\alpha]$-submodule $(\Pi^K_\rho)_\mu := \bigoplus_{\lambda \in \mathfrak{P}^{-1}_{L_1}(\mu)} (\Pi^K_\rho)_\lambda$, and a $k[\mathbb{Z} \cdot w^L_0(\alpha)]$-submodule $(\Pi^K_{w^L_0(\rho)})_\mu := \bigoplus_{\lambda \in \mathfrak{P}^{-1}_{L_2}(\mu)} (\Pi^K_{w^L_0(\rho)})_\lambda$, so that $I_{w^L_0}((\Pi^K_\rho)_\mu \otimes k([\alpha]_\rho)) \subset (\Pi^K_{w^L_0(\rho)})_\mu \otimes k([w^L_0(\alpha)]_{w^L_0(\rho)})$.

It follows that $R_{w^L_0}((\mathrm{End}_{L_1/L^c_1}\Pi^K_\rho)_\mu) \subset (\mathrm{End}_{L_2/L^c_2}\Pi^K_{w^L_0(\rho)})_\mu \otimes k([w^L_0(\alpha)]_{w^L_0(\rho)})$ where the $[\alpha]_\rho$ (respectively $[w^L_0(\alpha)]_{w^L_0(\rho)}$)-invariant Λ_L-grading on $\mathrm{End}_{L_1/L^c_1}\Pi^K_\rho$ (respectively on $\mathrm{End}_{L_2/L^c_2}\Pi^K_{w^L_0(\rho)}$) is induced by the grading on Π^K_ρ, $\Pi^K_{w^L_0(\rho)}$ as in (23).

Recall the following elementary fact. For a Laurent polynomial $P = \sum a_i t^i \in k[t, t^{-1}]$ set $\mathrm{supp}(P) = \{n \in \mathbb{Z} \,|\, a_n \neq 0\}$. Let $f \in k(t)$ be a rational function, and set $n_f = \max(\mathrm{ord}_0(f), -\mathrm{ord}_\infty(f))$. Then for $P \in k[t, t^{-1}]$ we have $fP \in k[t, t^{-1}] \Rightarrow \mathrm{supp}(fP) \subset [\inf(\mathrm{supp}(P)) - n_f, \sup(\mathrm{supp}(P)) + n_f]$.

Now notice that for $\mu \in \Lambda_L$: $\mathfrak{P}_{L_2}^{-1}(\mu) = w_0^L(\mathfrak{P}_{L_1}^{-1}(\mu))$, and recall that by Lemma 1.9 $R_{w_0^L}([\alpha]_\rho h) = [w_0^L(\alpha)]_{w_0^L(\rho)} R^L_{w_0}(h)$.

It is clear that $(\mathrm{End}_{L_1/L_1^c}\Pi_\rho^K)_\mu$ is a finite graded $k[\mathbb{Z}\alpha]$-module for $\mu \in \Lambda_L$. Hence the above elementary fact implies that at least for fixed μ there exists such n_0 that the condition b) of the Lemma holds for $h \in (\mathrm{End}_{L_1/L_1^c}\Pi_\rho^K)_\mu$. But if it holds for some h it also holds (with the same n_0) for $h' = l(h)$ where l is any element of A_L/A_L^c. Since a finite number of components of the Λ_L-grading generate $\mathrm{End}_{L_1/L_1^c}\Pi_\rho^K$ as an A_L/A_L^c-module, b) is proved. $\square$

1.14. The next statement implies Proposition 1.7.

Let us choose such $m \in \mathbb{Z}^{>0}$ that $\mathfrak{P}_L(mr) \in X_L$ for any L and any coroot r. In the notations of 1.13 choose α to be $\mathfrak{P}_{L_1}(mr)$.

Let $q_1, \dots, q_n \in k^\times$ be a set of elements such that $R^\rho_{w_0^L} \cdot (\prod([\alpha]_\rho - q_i)h)$ is regular for any ρ, L_1, L_2, L, w_0^L as in 1.13, α as above and $h \in \mathrm{End}_{L_1/L_1^c}(\Pi_\rho)$ (such a set exists by 1.13a)).

For $\lambda \in L_1/L_1^c$ denote

$$M^\rho_{x,\lambda} = m_x^\rho \cdot [\lambda]_\rho \prod_{r \in \Sigma^+ - \Sigma_L^+} \prod_{i=1}^{n} ([\mathfrak{P}_{L_1}(mr)]_\rho - q_i). \tag{26}$$

Proposition 1.15. *Let $L_2 = w(L_1) \neq L_1$, $L \supset L_1$ be standard Levi subgroups, $\rho \in Cusp_{L_1}$, and $x \in K\backslash G/P$.*

a) For any $\lambda \in L_1/L_1^c$ the element $R_w(M^\rho_{x,\lambda})$ is regular (i.e. lies in $\mathrm{End}(\Pi_{w(\rho)})$ rather than in $\mathrm{End}(\Pi_{w(\rho)}) \otimes Frac(k[L_2/L_2^c])$).

b) If $L_2 \subset L$ then $R_w\left(M^\rho_{x,\lambda}\right) = M^{w(\rho)}_{x,w(\lambda)}$. Moreover, if the integer m used in the definition of $M^\rho_{x,\lambda}$ is large enough, then $M^{w(\rho)}_{x,w(\lambda)} - m_x[\lambda]_{w(\rho)} \in \sum_{r \in \Sigma^+ - \Sigma+_L} \tilde{\mathcal{H}}_{\preccurlyeq \lambda + 2n\delta^L - ar}$.

c) If $\lambda \in X_L$ is dominant enough and $L_2 \not\subset L$, then $R_w\left(M^\rho_{x,\lambda}\right) \in \sum_{r \in \Sigma^+ - \Sigma+_L} \tilde{\mathcal{H}}_{\preccurlyeq \lambda + 2n\delta^L - ar}$, where $\delta^L = \frac{1}{2} \sum_{r \in \Sigma^+ - \Sigma_L^+} r$.

Proof b) The first statement follows directly from Lemma 1.11 and 1.9. To see the second notice that

$$\mathrm{supp}(R_w(M^\rho_{x,\lambda} - m_x[\lambda]_{w(\rho)})) = \{\lambda + \sum_{r \in \mathfrak{P}_{L_1}(\Sigma^+ - \Sigma_L^+)} i_r mr \mid 0 \leqslant i_r \leqslant n\} - \{\lambda + \sum_{r \in \mathfrak{P}_{L_1}(\Sigma^+ - \Sigma_L^+)} nmr\} \subset \bigcup_{r \in \mathfrak{P}_{L_1}(\Sigma^+ - \Sigma_L^+)} \mathfrak{a}_{\preccurlyeq \lambda + 2n\delta^L - m\mathfrak{P}_{L_1}(r)}$$

(Here we used that λ, $2\delta^L \in X_L \subset X_{L_1}$, so $\mathfrak{P}_{L_1}(\lambda + 2n\delta^L) = \lambda + 2n\delta^L$). It is well-known that $\mathfrak{P}_{L_1}$ preserves $\preccurlyeq$ (see e.g. [BW] Lemma 6.4 on p. 139, statement 2), so the statement is clear.

Let us prove a). Recall that the set of standard Levi conjugate to L_1 is in bijection with chambers into which the root hyperplanes divide $\mathfrak{a}_{L_1}$; the bijection sends a chamber C to $L_C = w_C(L_1)$, where $w_C \in W$ is an element such that $w(C) \in \mathfrak{a}^+$ (such an element is defined uniquely up to right multiplication by an element of W_L; we also have $w_C(C) = \mathfrak{a}^+_{L_C}$). Let $C_+ = \mathfrak{a}^+_L$, C be the chambers

corresponding respectively to L_1, L_2. On the set of chambers we have the length function $\ell(C) = \#\{r \in \mathfrak{P}_{L_1}(\Sigma^+) \,|\, \langle r, \alpha\rangle < 0 \text{ for } \alpha \in C\}$ (the latter condition will be abbreviated as $\langle r, C\rangle < 0$). We can choose a sequence of chambers $C_0 = C_+, C_1, \ldots, C_n = C$ so that C_i and C_{i+1} have a common face of codimension 1, and $\ell(C_i) = \ell(C_{i-1}) + 1$. Let $r_i \in \mathfrak{P}_{L_1}(\Sigma^+)$ be the (only) element such that $\langle r_i, C_{i-1}\rangle > 0$, $\langle r_i, C_i\rangle < 0$, and $L_i = L_{C_i}$. The sequence $C_1, \ldots C_n$ can be chosen so as to satisfy the following additional property:

$$r_i \in \mathfrak{P}_{L_1}(\Sigma_L^+) \Longrightarrow r_j \in \mathfrak{P}_{L_1}(\Sigma_L^+) \,\&\, L_j \subset L \text{ for all } j < i. \tag{27}$$

Indeed, such a sequence can be constructed inductively as follows. Suppose that $C_1, \ldots, C_i$ are already chosen, and that $L_1, \ldots, L_i \subset L$, $r_1, \ldots, r_i \in \mathfrak{P}_{L_1}(\Sigma_L^+)$. Assume first that there exists a coroot $r \in \Sigma_L$ separating C_i from C (i.e. $\langle r, \alpha\rangle \cdot \langle r, \beta\rangle < 0$ for $\alpha \in C_i$, $\beta \in C$). Then there exists also a coroot $r \in \Sigma_L^+$ which is orthogonal to a codimension 1 face of C_i, and separates C_i from C, for otherwise all simple roots of L are nonnegative on the cone $w_{C_i}(C)$, while some positive root of L is negative there, which is impossible. Any (co)root separating C_i from C separates also C_+ from C; since $r \in \Sigma_L^+$ we get $\langle r, C_+\rangle > 0$, $\langle r, C_i\rangle > 0$. We now take $r_{i+1} = \mathfrak{P}_{L_1}(r) \in \mathfrak{P}_{L_1}(\Sigma_L^+)$, and C_{i+1} to be the chamber neighboring with C_i and separated from C_i by r'. It is clear that $L_{i+1} \subset L$.

If none of the roots separating C_i from C lies in Σ_L, we choose the remaining $C_{i+1}, \ldots, C_n = C$ arbitrary (keeping only the requirements $\dim(C_i \cap C_{i+1}) = \dim(C_i) - 1$, $\ell(C_{i+1}) = \ell(C_i) + 1$). Since r_i separates C_j from C for $j < i$ we see that the resulting sequence of chambers satisfies (27).

Fix $(C_1, \ldots, C_n)$ satisfying (27); let l be the smallest integer for which $r_j \notin \Sigma_L^+$. We can write:

$$\begin{gathered}
R_w(M_{x,\lambda}^\rho) = (R_{w_C \cdot w_{C_{n-1}}^{-1}}) \circ (R_{w_{C_{n-1}} \cdot w_{C_{n-2}}^{-1}}) \circ \ldots (R_{w_{C_{l+1}} \cdot w_{C_l}^{-1}}) \circ R_{w_{C_l}}(M_{x,\lambda}^\rho) = \\
(R_{w_C \cdot w_{C_{n-1}}^{-1}}) \circ (R_{w_{C_{n-1}} \cdot w_{C_{n-2}}^{-1}}) \circ \cdots \circ (R_{w_{C_{l+1}} \cdot w_{C_l}^{-1}})(M_{x, w_{C_l}(\lambda)}^{w_{C_l}(\rho)}) = \\
\left(R_{w_{C_n} \cdot w_{C_{n-1}}^{-1}} \prod([w_{C_{n-1}}(mr_n)]_{w_{C_{n-1}}(\rho)} - q_j)\right) \circ \ldots \\
\circ \left(R_{w_{C_{l+1}} \cdot w_{C_l}^{-1}} \prod([w_{C_l}(mr_{l+1})]_{w_{C_l}(\rho)} - q_j)\right) \\
\left([w_{C_l}(\lambda)]_{w_{C_l}(\rho)} m_x^{w_{C_l}(\rho)} \prod_{r \in \mathfrak{P}_{L_1}(\Sigma^+ - \Sigma_L^+), r \neq r_i} \prod_{j=1}^n ([mr]_{w_{C_l}(\rho)} - q_j)\right).
\end{gathered} \tag{28}$$

Notice that L_{i-1}, L_i generate a standard Levi L^i of rank equal to $\operatorname{rank}(L_i) + 1$, $w_{C_i} w_{C_{i-1}}^{-1} \in w_0^{L^i} \cdot W_{L_{i-1}}$ and $w_{C_{i-1}}(r_i)$ is the only simple root of L^i which is not a root of L_{i-1}. Hence, by the choice of $q_1, \ldots, q_n$, each of the operators $R_{w_{C_i} \cdot w_{C_{i-1}}^{-1}} \prod([w_{C_{i-1}}(mr_i)]_{w_{C_{i-1}}(\rho)} - q_j)$ is regular, and thus the whole expression (28) is regular.

Let us prove c). We show that for $C_1, \ldots, C_n$, l as above and $i > l$

$$\mu \in \operatorname{supp}(w_{C_i}(M_{x,\lambda}^\rho)) \Rightarrow \exists \nu \in \operatorname{supp}(w_{C_{i-1}}(M_{x,\lambda}^\rho)),\ r \in \Sigma^+ - \Sigma_L^+ :\ \mu \preccurlyeq \nu - ar.$$

Recall that $w_{C_{i-1}}w_{C_i}^{-1} \in w_0^{L^i} \cdot W_{L_i}$; also the vector $\alpha := w_{C_{i-1}}(r_i) \in X_{L_i}$ is orthogonal to X_{L^i}. Thus by Lemma 1.13b) there exists $\nu \in \operatorname{supp}(w_{C_{i-1}}(M^\rho_{x,\lambda}))$ such that $\nu = w_0^{L^i}(\mu) + N\alpha$, where $|N| < n_0$. Decompose ν as $\nu = \mathfrak{P}_{L^i}(\nu) + \frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$.

Let $\tilde{r}_i \in \Sigma^+$ be a coroot such that $\mathfrak{P}_{L_1}(\tilde{r}_i) = r_i$. Then $w_{C_{i-1}}(\tilde{r}_i) \succ 0 \succ w_{C_i}(\tilde{r})$, because (the dual root to) $w_{C_{i-1}}(\tilde{r}_i)$ is positive on $w_{C_{i-1}}(C_{i-1}) \subset \mathfrak{a}^+$, while (the dual root to) $w_{C_i}(\tilde{r})$ is negative on $w_{C_i}(C_i) \subset \mathfrak{a}^+$.

Since the orthogonal projection $\mathfrak{P}_L$ preserves $\preccurlyeq$, we have: $\alpha = \mathfrak{P}_{L_{i-1}}(w_{C_{i-1}}(\tilde{r}_i)) \succ 0 \succ w_0^{L^i}(\alpha) = (w_0^{L^i})^{-1}(\alpha)\mathfrak{P}_{L_i}(w_{C_i}(\tilde{r}_i))$. Moreover, since α belongs to a fixed finite subset of $\mathfrak{a}^+_{L_1}$, there exists $b > 0$ and $r \in \Sigma_+ - \Sigma_+^L$ such that $\alpha \succcurlyeq br$.

To finish the proof it is enough to show that $\langle\nu,\alpha\rangle \gg 0$ if $\lambda \in X_L$ is very dominant; in fact it is enough to ensure that

$$\langle\nu,\alpha\rangle > \left(\frac{a}{b} + n_0\right)\langle\alpha,\alpha\rangle, \tag{29}$$

for then we get $\mu = (w_0^{L^i})^{-1}(\nu - N\alpha) = \mathfrak{P}_{L^i}(\nu) + \left(\frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle} - N\right) w_0^{L^i}(\alpha) \prec \mathfrak{P}_{L^i}(\nu) \prec \mathfrak{P}_{L^i}(\nu) + \frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha \preccurlyeq \nu - ar$.

To check (29) notice that from 1.13b) it follows by induction on i that for $w = w_{C_i}$ there exists a finite set S such that $\operatorname{supp}(R_w(M^\rho_{x,\lambda}))$ is contained in the convex hull of $w(\operatorname{supp}(M^\rho_{x,\lambda})) + S$ for any λ, ρ (where we used the notation $A + B = \{a + b | a \in A, b \in B\}$). Hence

$$\min_{\nu \in \operatorname{supp}(R_{w_{C_{i-1}}}(M^\rho_{x,\lambda}))}(\langle\nu,\alpha\rangle) \leqslant \min_{\nu \in w_{C_{i-1}}(\operatorname{supp}(M^\rho_{x,\lambda}))}(\langle\nu,\alpha\rangle) + \text{const}\,.$$

Since $\alpha = w_{C_{i-1}}(r_i)$ we have $\min\limits_{\nu \in w_{C_{i-1}}(\operatorname{supp}(M^\rho_{x,\lambda}))}\langle\nu,\alpha\rangle = \min\limits_{\nu \in \operatorname{supp}(M^\rho_{x,\lambda})}\langle\nu,r_i\rangle \gg 0$ if $\nu \in X_L$ is very dominant, because our assumption $i > l$ implies that $r_i \in \Sigma^+ - \Sigma^+_L$. This proves (29). $\square$

1.16. To deduce Proposition 1.7 from 1.15 we take

$$h^\rho_{x,\lambda} := \sum_{L_2 = w(L_1)} R_w(M^\rho_{x,\lambda}). \tag{30}$$

From Proposition 1.15 it follows that $h^\rho_{x,\lambda} \in \tilde{\mathcal{H}}$ and that $h^\rho_{x,\lambda} - \sum\limits_{L_2 = w(L_1) \subset L}[\lambda]_{w(\rho)} m_x^{w(\rho)} \in \sum\limits_{r \in \Sigma^+ - \Sigma^+_L} \tilde{\mathcal{H}}_{\preccurlyeq \lambda + 2n\delta^L - ar}$. It is obvious that $h^\rho_{x,\lambda}$ is invariant under all the intertwining operators R_w (more precisely $R_w(h_{\rho'}) = h_{w(\rho')}$ where $h^\rho_{x,\lambda} = \sum_{Cusp^K} h_{\rho'}$, $h_{\rho'} \in \operatorname{End}(\Pi_{\rho'})$). Also it is immediate to check that $T_\psi(m^\rho_x) = m^\rho_x$, $R_w \circ T_\psi \circ R_w^{-1} = T_{w(\psi)}$ when T_ψ is defined. If $\rho \otimes \psi \cong \psi$ for some irreducible representation ρ of a standard Levi L, then $\psi|_{\iota_L(X_L)} = 1$, hence for $\lambda \in X_L$ we have $T_\psi([\lambda]_\rho) = [\lambda]_{\rho\otimes\psi}$.

This implies that each summand in (30) is invariant under T_ψ when the latter is defined, provided $\lambda \in X_L$. By the matrix Payley-Wiener Theorem we conclude that $h^\rho_{x,\lambda} \in I(\mathcal{H})$. Since $2\delta^L \in X_L$ the coweight $\lambda + 2n\delta^L$ runs over the set of very dominant elements of X_L when λ does. Proposition 1.7 is proved. $\square$

Corollary 1.17. *Set*

$$h_x^\lambda := \sum_{L_1 \subset L, \rho \in Cusp_{L_1}^K} \frac{1}{\#\{L' \subset L | L' \in W_L(L_1)\}} h_{x,\lambda}^\rho.$$

Then we have

$$I(h_x^\lambda) - [\lambda] m_x \in \sum_{r \in \Sigma^+ - \Sigma_L^+} \tilde{\mathcal{H}}_{\preccurlyeq \lambda - a \cdot r}$$

provided $\lambda \in X_L^+$ *is dominant enough, and the integer* m *used in* (26) *is large.* □

1.18. We next want to consider associated graded objects, so we have to pass to a complete order.

Let $\leqslant$ be any complete order on X strengthening $\preccurlyeq$. It yields filtrations on $\mathcal{H}$, $\tilde{\mathcal{H}}$ as in 1.1–1.6.

We write $gr\mathcal{H}$ for the associated graded algebra of $\mathcal{H}$, $\leqslant$, and use the letter I for the induced imbedding $gr\mathcal{H} \hookrightarrow gr\tilde{\mathcal{H}} \cong \tilde{\mathcal{H}}$.

We have an obvious map $K_0 \to gr_0\mathcal{H}$.

Theorem 1.19. *a)* $gr\mathcal{H}$ *is Noetherian.*

b) Let M *be any finitely generated graded module over* $gr\mathcal{H}$*. Then for all but finitely many* $\lambda \in X$ *we have* $Tr(g, M_\lambda) = 0$.

Proof Let Z be the center of $\mathcal{H}$.

The explicit description of Z (Theorem 2.13 of [BD]) implies that the following elements form a basis in $I(grZ) \subset gr\tilde{\mathcal{H}} \cong \tilde{\mathcal{H}}$. For a standard Levi L, $\rho \in Cusp_L^K$, $\lambda \in (L/L^c)^+$ such that $\psi(\lambda) = 1$ whenever $\psi \otimes \rho \cong \rho$ denote $z_\lambda^\rho := \sum\limits_{w(\lambda)=\lambda} [\lambda]_{w(\rho)\otimes\psi_{w(\rho)}}$, where w runs over the set of such $w \in W/W_L$ for which $w(L)$ is a standard parabolic and $w(\lambda) = \lambda$.

It follows that for $\lambda \in X^+$ we have $[\lambda] \in grZ$.

It is clear that grZ is central in grH, and (as we will soon see) it is of finite type.

Thus to prove a) it suffices to show that $gr\mathcal{H}$ is a finite grZ module. The main step is the next:

Lemma 1.20. *There exists* $\lambda_0 \in \mathfrak{a}$ *such that* $gr\mathcal{H}_\lambda \neq 0 \Rightarrow \lambda \in \lambda_0 + \mathfrak{a}^+$.

Proof of the Lemma. Assume the contrary. Then for any $N > 0$ we can find $\lambda \in X$ such that $\langle \lambda, r \rangle < -N$ for a simple coroot r, and $grH_\lambda \neq 0$. Let λ be like that, then there exists $h \in \mathcal{H}$ such that $\lambda \in \mathrm{supp}(h) \subset X_{\leqslant\lambda}$. Decompose $h = \sum h_\rho$, $h_\rho \in \mathrm{End}(\Pi_\rho)$, and fix $\rho \in Cusp_L^K$ for which $\mathrm{supp}(h_\rho) \ni \lambda$. Since $\langle r, \lambda \rangle \neq 0$ we have $r \notin \Sigma_L$. Let L' be the standard Levi such that $L' \supset L$, $\mathrm{rank}(L') = \mathrm{rank}(L) + 1$ and $r \in \Sigma_{L'}$. Then writing $\lambda = \mathfrak{P}_{L'}(\lambda) + \frac{\langle \lambda, r \rangle}{\langle \mathfrak{P}_L(r), \mathfrak{P}_L(r) \rangle} \mathfrak{P}_L(r)$ we get by 1.13b) an element $\nu \in \mathrm{supp}(R_{w_0^{L'}}(h)) = \mathrm{supp}(h)$ such that $\nu = w_0^{L'}(\lambda) + N' w_0^{L'}(\mathfrak{P}_L(r)) = \mathfrak{P}_{L'}(\lambda) + \left(\frac{\langle \lambda, r \rangle}{\langle \mathfrak{P}_L(r), \mathfrak{P}_L(r) \rangle} + N' \right) w_0^{L'}(\mathfrak{P}_L(r))$, where $|N'| < n_0$. We have $\mathfrak{P}_L(r) \succ 0$, $w_0^{L'}(\mathfrak{P}_L(r)) = \mathfrak{P}_{w_0^{L'}(L)}(w_0^{L'}(r)) \prec 0$, hence taking $N > n_0 \cdot \langle \mathfrak{P}_L(r), \mathfrak{P}_L(r) \rangle$ we obtain $\nu \succ \mathfrak{P}_{L'}(\lambda) \succ \lambda \Rightarrow \nu > \lambda$. This contradicts the condition $\mathrm{supp}(h) \subset \mathfrak{a}_{\leqslant\lambda}$.□

Lemma 1.21. *For any* L *fix* $\lambda_L \in X_L^+$*. The subalgebra* $grZ_L \subset grZ$ *generated by* $[\lambda]$ *for* $\lambda \in \lambda_{L'} + X_{L'}^+$*,* $L' \subseteq L$ *is of finite type. The algebra* $\mathcal{Z}_L :=$

$\bigoplus_{L',\rho\in Cusp^K_{L'}} \bigoplus_{\lambda\in(L'/(L')^c)\cap\mathfrak{a}^+_L} k\cdot[\lambda]_\rho \subset \tilde{\mathcal{H}}$, *where* L' *runs over the set of standard Levi, is a finite* grZ_L*-module.*

Proof follows from the next easy fact. Fix $n\in\mathbb{Z}_{\geqslant 0}$ and consider the subsemigroup in $\mathbb{Z}^m_{\geqslant 0}$ generated by $\mathbb{Z}^I_{\geqslant n}$, where I runs over the subsets of $[1,m]$. This semigroup is finitely generated, and $\mathbb{Z}^m_{\geqslant 0}$ is the union of a finite number of cosets of this semigroup. □

It is clear that $\bigoplus_{\lambda\in\lambda_0+\mathfrak{a}_+} gr\tilde{\mathcal{H}}_\lambda$ is a finite module over $\mathcal{Z}_G$. Hence by 1.21 it is also finite over grZ_G. By 1.20 we have $gr\mathcal{H}\subset \bigoplus_{\lambda\in\lambda_0+\mathfrak{a}_+} gr\tilde{\mathcal{H}}_\lambda$ for some λ_0 and by 1.21 grZ_G is Noetherian, hence $gr(H)$ is finite over grZ_G. a) is proved. Notice that along the way we have shown also that $gr\mathcal{H}$ is finite over $grZ\supset grZ_G$.

Proof of b). Consider a subalgebra $\mathcal{S}_L\subset gr\mathcal{H}$ generated by the highest terms of the elements h^λ_x (see 1.17), with large $\lambda\in X^+_L$, i.e. by the elements $[\lambda]m_x$.

Notice that $\mathcal{S}_L\supset grZ_L$ (for appropriate choice of coweights λ_L in 1.21), because $\sum_{x\in K\backslash G/P}[\lambda]m_x=[\lambda]$. Hence $gr\mathcal{H}_L$ is finite over $\mathcal{S}_L$, and thus any finite $gr\mathcal{H}_L$-module is also finite over $\mathcal{S}_L$.

We prove by induction on corank of L, that for any graded finitely generated $\mathcal{S}$-module M we have $Tr(g,M_\lambda)=0$ for all but finitely many $\lambda\in X_L$. The step of induction is as follows.

Let M be an X_L-graded finitely generated $\mathcal{S}_L$-module.

By the definition $\mathcal{S}_L$ contains for some $\lambda_L\in X^+_L$ the subalgebra $\mathcal{A}_L[\lambda_0+X^+_L]$, where

$$\mathcal{A}_L:=\bigoplus_{x\in K\backslash G/P} k\cdot m_x. \tag{31}$$

Hence the proof of the Hilbert Basis Theorem implies that for any graded finitely generated $\mathcal{S}_L$-module M there exists μ_0 and a finite $\mathcal{A}_L$-module M_0 such that $M_\mu\cong M_0$ for $\mu\in\mu_0+X^+_L$, this isomorphism being compatible with the action. In other words we have an isomorphism of graded $\mathcal{A}_L[\lambda_0+X^+_L]$-modules $\bigoplus_{\mu\in\mu_0+X^+_L} M_\mu\cong M_0[\mu_0+X^+_L]$.

Further, for $\lambda\in\lambda_0+X^+_L$, $M_\lambda\cong M_0$ is the direct sum of the images of idempotents $m_x\in\mathcal{A}_L$, and we have $g(\mathrm{Im}(m_x))=\mathrm{Im}(m_{g(x)})$. We have assumed that g acts on $K\backslash G/P$ without fixed points, hence $tr(g,M_\lambda)=0$.

It is obvious that $\Lambda-\cup(\lambda_0+X^+_L)$ is a finite union of cosets $\mu_i+X^+_{L_i}$, $L_i\supset L$. Then $\bigoplus_{\lambda\in\mu_i+X^+_{L_i}} M_\lambda$ is a finitely generated graded module over $\mathcal{S}_{L_i}$, because it is contained in $\bigoplus_{\lambda\in\mu_i+\mathfrak{a}^+_{L_i}} M_\lambda$ which is finite over grZ_λ. Hence the statement for M follows from the induction hypothesis. The Theorem is proved. □

Now for any finitely generated graded module N over $gr\mathcal{H}$ we can define $Tr(g,N):=\sum_\lambda Tr(g,N_\lambda)$ the sum being finite by the last Theorem.

Let M be a finitely generated $\mathcal{H}$-module.

1.22. We can choose an $(\mathfrak{a}, \leqslant)$-filtration F on M compatible with $\leqslant$ so that associated graded module $gr_F M$ is a finite $gr\mathcal{H}$-module (such a filtration is called a good filtration).

1.23. *From now on we assume that the complete order $\leqslant$ is induced by the lexicographic order under the isomorphism $\mathfrak{a} \cong \mathbb{R}^r$, $\lambda \mapsto ((e_1, \lambda), ..., (e_r, \lambda))$, where $(e_1, \ldots, e_r)$ is a basis in $(X \otimes \mathbb{Q})^*$, such that $(e_i, r) \geqslant 0$ for $r \in \Sigma^+$.*

Proposition 1.24. *a) $Tr(g, gr_F M)$ does not depend on F.*
We set $Tr(g, M) := Tr(g, gr_F M)$. Then $Tr(g, M)$ satisfies:
b) 0.23i), and
c) 0.23ii).

Proof Recall the following:

Fact 1.25. *Let H be an algebra with an increasing $\mathbb{Z}$-filtration bounded below (i.e. $F_{\leqslant n} = 0$ for $n \ll 0$), such that the associated graded algebra is Noetherian; let M be a finite H-module.*

Let $gr(H)$-Mod be the category whose objects are finitely generated graded $gr(H)$-modules and morphisms are homomorphisms of modules preserving the degree up to a shift (i.e. $Mor(M, N) = \sum Mor(M, N)_\nu$ where $Mor(M, N)_\nu = \{\phi : M \to N \mid \phi(M_\lambda) \subset N_{\lambda+\nu}$). Then the class of grM in the Grothendieck group $K^0(gr(H)$-Mod) does not depend on the choice of a good filtration on M.

Thus $[M] \to [grM]$ yields a well-defined homomorphism from $K^0(H$-Mod) to $K^0(grH$-Mod).

Proof : see e.g. [CG] Corollary 2.3.19 on p. 79. (The authors of *loc. cit.* consider the Grothendieck group of the category of finitely generated grH-modules without grading; however their argument actually proves the above statement as well). $\square$

We apply 1.25 inductively.

Let the sublattice $\Lambda_0 \subset \mathfrak{a}$ be generated by Λ_L for all L. It is obvious that the set of indices λ for which $gr\mathcal{H}_\lambda \neq 0$ is contained in Λ_0.

For $i \in [1, .., n]$ let $\Lambda^{(i)}$ be the image of Λ_0 in $\mathbb{Q}^i$ under $pr_i : \lambda \to ((\lambda, e_1), \ldots, (\lambda, e_i))$; it is convenient to choose e_i so that $\Lambda^{(i)} \subset \mathbb{Z}^i$. Equip $\Lambda^{(i)}$ with the lexicographic order $\leqslant$ induced from $\mathbb{Z}^i$. We have a $\Lambda^{(i)}$-filtration $F^{(i)}_{\leqslant\overline{\lambda}}(\mathcal{H}) = \bigcup\limits_{\lambda, pr_i(\lambda) = \overline{\lambda}} \mathcal{H}_{\leqslant\lambda}$ on $\mathcal{H}$; notice that $F^{(i)}$ is induced by the corresponding $\Lambda^{(i)}$-filtration on $\tilde{\mathcal{H}}$.

Furthermore, consider the one parameter filtration $\tilde{\mathcal{H}}$ given by $\tilde{\mathcal{H}}_{\underset{i+1}{\leqslant} n} = \bigoplus\limits_{(\lambda, e_i) \leqslant n} \tilde{\mathcal{H}}_\lambda$, and the induced filtration on $\mathcal{H}$ denoted by the same symbol.

It yields a filtration on $gr_{F^{(i)}}\mathcal{H}$ given by the rule: $gr_{F^{(i)}}\mathcal{H}_{\underset{i+1}{\leqslant} n} = \bigoplus\limits_{\overline{\lambda} \in X^{(i)}} [gr_{F^{(i)}}\mathcal{H}_{\overline{\lambda}}]_{\underset{i+1}{\leqslant} n}$ where $[gr_{F^{(i)}}\mathcal{H}_{\overline{\lambda}}]_{\underset{i+1}{\leqslant} n} = F^{(i)}_{\leqslant\overline{\lambda}}(\mathcal{H}) \cap \mathcal{H}_{\underset{i+1}{\leqslant} n} / F^{(i)}_{<\overline{\lambda}}(\mathcal{H}) \cap \mathcal{H}_{\underset{i+1}{\leqslant} n}$.

Notice also that this filtration on $gr_{F^{(i)}}\mathcal{H}$ is induced by the $\underset{i+1}{\leqslant}$-filtration on $\tilde{\mathcal{H}} \cong gr_{F^{(i)}}(\tilde{\mathcal{H}})$ under the imbedding $gr_{F^{(i)}}(I)$.

We see that $gr_{F^{(i+1)}}\mathcal{H} = gr_{\underset{i+1}{\leqslant}}[gr_{F^{(i)}}\mathcal{H}]$.

If now a good X-filtration F on an $\mathcal{H}$-module M is given, then again we can form $\Lambda^{(i)}$-filtrations $F^{(i)}(M)$, where $F^{(i)}_{\leqslant\overline{\lambda}}(M) = \bigcup\limits_{\lambda, pr_i(\lambda)=\overline{\lambda}} M_{\leqslant\lambda}$. Consider the associated graded $gr_{F^{(i)}}M$; on $gr_{F^{(i)}}M$ we have a one parameter filtration denoted by $\underset{i+1}{\leqslant}$, and given by $[gr_{F^{(i)}}M_{\overline{\lambda}}]_{\underset{i+1}{\leqslant} n} = \bigcup\limits_{pr_i(\lambda)=\overline{\lambda}, (e_{i+1},\lambda)\leqslant n} M_{\leqslant\lambda} / \bigcup\limits_{pr_i(\lambda)<\overline{\lambda}, (e_{i+1},\lambda)\leqslant n} M_{\leqslant\lambda}$.

Obviously, $gr_{F^{(i+1)}}M = gr_{\underset{i+1}{\leqslant}}[gr_{F^{(i)}}M]$.

By inverse induction in i we see that $\underset{i+1}{\leqslant}$ is good; hence using 1.25 and induction in i we get the following statement:

The class of $gr^{(i)}_F M$ in the Grothendieck group of the category of $\Lambda^{(i)}$-graded $gr^{(i)}_F\mathcal{H}$-modules, with morphisms preserving grading up to a shift, is independent on F.

(Actually to make a step of induction we need a variant of 1.25 applicable in the situation when H and M have some multi-grading respected by the filtration while grH and grM are considered as multi-graded objects with one additional index. One may say that what we need is exactly the statement 1.25 but with algebras/modules being algebras/modules in the category of multigraded vector spaces rather than in the category of vector spaces).

a), b) follow from the last italicized statement directly. c) is obvious: if $\mathfrak{M}$ is an admissible representation, and $M = \mathfrak{M}^K$ then for large λ we have:

$$Tr(g, M) = tr(g, F_{\leqslant\lambda}(M)) = tr(g, M) = tr(\delta_K \cdot g, \mathfrak{M}) = \int_{g'\in K} \chi_{\mathfrak{M}}(g'g)dg' = \chi_{\mathfrak{M}}(g)$$

provided the integrand is constant. □

2. **Geometric filtration and asymptotic cones.** In 1.24 we have defined a functional $Tr(g, M)$ satisfying property i), ii) of Theorem 0.23; it remains to prove that it satisfies also property iii). This will be accomplished in the next two sections by passing from the "spectral" filtration constructed above to a "geometric" one defined in terms of support of a distribution $h \in \mathcal{H}$; the latter filtration is directly related to orbital integrals.

The main results of this section used in further arguments are Theorem 2.7, Proposition 2.12, and Theorem 2.15.

2.1. We can assume that K is *nice* with respect to L_0 (nice in the terminology of [BD] 2.1b), i.e. that for any pair of opposite parabolics $P = L \cdot U, P^- = L \cdot U^-$, with $L \supset L_0$ standard we have $K = K^+ \cdot K^0 \cdot K^-$ where as usual $K^+ = U \cap K$, $K^- = U^- \cap K$, $K^0 = L \cap K$. (More precisely, the condition that K_0, L_0 are in good relative position implies that there exists a K_0-stable lattice $\mathfrak{g} \subset \mathrm{Lie}\, G$ such that $\mathfrak{g} = (\mathfrak{g} \cap \mathrm{Lie}\, U^-) \oplus (\mathfrak{g} \cap \mathrm{Lie}\, L) \oplus (\mathfrak{g} \cap \mathrm{Lie}\, U)$; we can take $\mathfrak{g}$ to be the Lie algebra of the O-algebraic group attached to the K_0-fixed point p of the building by the Bruhat-Tits theory [BT2]. We then obtain K from $\mathfrak{p}^N\mathfrak{g}$ for large N as in [BD]2.1b)).

Fix opposite parabolics $P = L \cdot U$, $P^- = L \cdot U^-$ with P, L standard, and set $C_P = (G/U \times G/U^-)/L$. By $\mathcal{H}(C_P)$ we denote the space of K-biinvariant compactly supported measures on C_P.

Consider the imbedding $\psi : L \hookrightarrow C_P$ given by $\psi : l \mapsto (l \cdot U, 1 \cdot U^-) \mod L$.

Let $T \subseteq L_0$ be the (unique) maximal split torus contained in L_0. For $\lambda \in X$ let $T_\lambda \in T/T^c$ be the coset of λ. (Recall that we have an imbedding $X \hookrightarrow T$, $\chi \mapsto \chi(\mathfrak{p})$).

Proposition 2.2. *Let $\mathfrak{C} \subset G$ be a compact K-biinvariant set. For $\lambda \in X^+$ such that $\langle \lambda, \Sigma^+ - \Sigma^+_L \rangle \gg 0$, and $t \in T_\lambda$, there exists a unique bijection $\Psi_{\mathfrak{C},t} : K\backslash \mathfrak{C} \cdot t \cdot \mathfrak{C}/K \to K\backslash \mathfrak{C} \cdot \psi(t) \cdot \mathfrak{C}/K$ which satisfies $\Psi_{\mathfrak{C},t}(K \cdot c_1 t c_2 \cdot K) = K \cdot c_1 \psi(t) c_2 \cdot K$ for $c_1, c_2 \in \mathfrak{C}$.*

Proof We have only to check that

$$K \cdot g_1 t g_2 \cdot K = K \cdot g_1' t g_2' \cdot K \Leftrightarrow K \cdot g_1 \psi(t) g_2 \cdot K = K \cdot g_1' \psi(t) g_2' \cdot K \quad \text{for } g_1, g_2, g_1', g_2' \in \mathfrak{C}.$$

Set $\mathfrak{C}' = \{g_1 g_2^{-1} | g_1, g_2 \in \mathfrak{C}\} \cup \{g_2 g_1^{-1} | g_1, g_2 \in \mathfrak{C}\}$, and $K' := \bigcap\limits_{g \in \mathfrak{C}'} gKg^{-1}$ for $g \in \mathfrak{C}$. It is clearly enough to verify that

$$g_1 \psi(t) g_2 = \psi(t),\ g_1, g_2 \in \mathfrak{C}' \Longrightarrow g_1 t g_2 \in K' \cdot t \cdot K' \tag{32}$$

$$g_1 t g_2 = t,\ g_1, g_2 \in \mathfrak{C}' \Longrightarrow g_1 \psi(t) g_2 \in K' \cdot \psi(t) \cdot K' \tag{33}$$

If equality in the LHS of (32) holds then we have: $g_1 = lu$, $g_2 = t^{-1} l t u^-$ where $u \in U$, $u^- \in U^-$, $l \in L$. Since g_1, g_2 lie in the compact set $\mathfrak{C}'$, the elements l, $t^{-1} l t$, u, u^- are in some bounded subsets of L, U, U^-. Hence the condition $\langle \lambda, \Sigma^+ - \Sigma^+_L \rangle \gg 0$ implies that $(tg_2)^{-1} u (tg_2) \in K$, $(lt) u^- (lt)^{-1} \in K$, which yields $g_1 t g_2^{-1} = [(lt) u_- (lt)^{-1}]^{-1} t [t^{-1} u t] \in KtK$.

If equality in the LHS of (33) holds, then $g_1 = t g_2 t^{-1} \in \mathfrak{C}' \cap t \mathfrak{C}' t^{-1}$. It is well-known that for $t \in T_\mu$, $\langle \mu, \Sigma^+ - \Sigma^+_L \rangle \gg 0$ we have $\mathfrak{C}' \cap t \mathfrak{C}' t^{-1} \subset \mathfrak{C}^- \cdot \mathfrak{C}^L \cdot \mathfrak{C}^+$, where $\mathfrak{C}^+$, $\mathfrak{C}^L$ are bounded (independently of t) subsets of respectively U and L, and $\mathfrak{C}^-$ is an arbitrary small subset of U^-; in particular we can assume $\mathfrak{C}^- \subset (K')^-$. Thus we have $g_1 = u_- \cdot l \cdot u$ with $u_- \in K^-$, $l \in \mathfrak{C}^L$, $u \in \mathfrak{C}^+$. Then $g_2 = t^{-1} g_1 t$, and we can assume $t^{-1} u t \in (K')^+$. So finally we get $(g_1, g_2)(\psi(t)) = (u_-, t^{-1} u t)(\psi(t)) \in K' \times K'(\psi(t))$. □

If $t \in T_\mu$ where $\langle \mu, \alpha \rangle$ is so large for $\alpha \in \Sigma^+ - \Sigma^+_L$ that $\Psi_{\mathfrak{C}'',t}$ is defined for $t \in T_\mu$ and $\mathfrak{C}'' := \{g_1 g_2^{-1} g_3 \,|\, g_1, g_2, g_3 \in \mathfrak{C}\}$, then obviously $\Psi_{\mathfrak{C},t}$ and $\Psi_{\mathfrak{C},t'}$ agree on the intersection $\mathfrak{C} t \mathfrak{C} \cap \mathfrak{C} t' \mathfrak{C}$ whenever this intersection is nonempty. Thus we have a bijection $\Psi : K\backslash (\bigcup\limits_{\langle \lambda, \Sigma^+ - \Sigma^+_L \rangle \gg 0} \mathfrak{C} \cdot T_\lambda \cdot \mathfrak{C})/K \to K\backslash (\bigcup\limits_{\langle \lambda, \Sigma^+ - \Sigma^+_L \rangle \gg 0} \mathfrak{C} \cdot \psi(T_\lambda) \cdot \mathfrak{C})/K$.

2.3. The map $w \mapsto K_0 \cdot w \cdot K_0$ yields a canonical bijection

$$W_p \backslash W_{aff} / W_p \widetilde{\longrightarrow} K_0 \backslash G / K_0,$$

see [BT1] 7.4.15 (notice that since K_0 and L_0 are in good relative position we have $K_0 \supset L_0^c$, so the expression $K_0 \cdot w \cdot K_0$ is meaningful).

Consider the projection $pr_L : W_p \backslash W_{aff} / W_p \to W_f^p \backslash W_{aff}^p / W_f^p = \Lambda / W_f^p = \Lambda^+$. For $x \in W_p \backslash W_{aff} / W_p$ we write G_x for the corresponding double coset in G, $\mathcal{H}_x$ for the space of elements in $\mathcal{H}$ supported in G_x, and we set $G_\lambda := \bigcup\limits_{pr_L(x) = \lambda} G_x$, $\mathcal{H}_\lambda := \bigoplus\limits_{pr_L(x) = \lambda} \mathcal{H}_x$.

(Support of an element $h \in \mathcal{H}$ appearing here should not be confused with support of the corresponding element $I(h) \in \tilde{\mathcal{H}}$ considered before (see (25)): the

first one is an open compact subset of G while the second is a finite subset of X. To rule out the very possibility of such confusion we will sometimes refer to the first notion as to the *geometric support*, and to the second one as to the *spectral support* and write respectively supp^{spec} and supp^{geom}).

Let us choose $\mathfrak{C}$ to be large enough; more precisely, we require the following. We can choose a finite number of elements $w_i \in W_{aff}$, so that for any $w \in W_{aff}$ there exist i_1, i_2 such that $w_{i_1} w w_{i_2} \in X^+$. Let us take $\mathfrak{C}$ to be $\bigcup_i (K_0 \cdot w_i) \cup \bigcup_i (w_i K_0)$. Then we have $\bigcup_{\langle \lambda, \Sigma^+ - \Sigma_L^+ \rangle \gg 0} \mathfrak{C} \cdot T_\lambda \cdot \mathfrak{C} \supset \bigcup_{\mu \in \mathfrak{a}^+, \langle \mu, \Sigma^+ - \Sigma_L^+ \rangle \gg 0} G_\mu$.

Denote $(C_P)_\mu := \bigcup_{g \in G_\mu} \Psi(KgK)$, $\mathcal{H}(C_P)_\mu = \{f \in \mathcal{H}(C_P) \,|\, \mathrm{supp}\, f \subset (C_P)_\mu\}$.

Thus we have a bijection

$$\bigcup_{\mu \in \mathfrak{a}^+, \langle \mu, \Sigma^+ - \Sigma_L^+ \rangle \gg 0} K \backslash G_\mu / K \widetilde{\longrightarrow} \bigcup_{\mu \in \mathfrak{a}^+, \langle \mu, \Sigma^+ - \Sigma_L^+ \rangle \gg 0} K \backslash C_\mu / K.$$

Let us set

$$\mathcal{H}_N^P = \bigoplus_{\langle \mu, \alpha \rangle \geqslant N, \alpha \in \Sigma^+ - \Sigma_L^+} \mathcal{H}_\mu, \quad \mathcal{H}(C_P)_N = \bigoplus_{\langle \mu, \alpha \rangle \geqslant N, \alpha \in \Sigma^+ - \Sigma_L^+} \mathcal{H}(C_P)_\mu,$$

and use the notation Ψ for the isomorphism $\mathcal{H}_N^P \widetilde{\longrightarrow} \mathcal{H}(C_P)_N$ sending the δ-function of a double K-coset x on G to the δ-function of $\Psi(x)$. (Here $N \gg 0$).

Lemma 2.4. *Let $\mathfrak{C} \subset G$ be compact. Let N be so large that $\Psi_{\mathfrak{C} \cdot \mathfrak{C}, t}$ is defined for $\langle \mu, \Sigma^+ - \Sigma_L^+ \rangle \geqslant N$, $t \in T_\mu$ where $\mathfrak{C} \cdot \mathfrak{C} = \{c_1 c_2 |\, c_1, c_2 \in \mathfrak{C}\}$. Then we have* $\mathrm{supp}(h) \subset \mathfrak{C}$, $h' \in \mathcal{H}_N^P \Rightarrow \Psi(h * h')$, $\Psi(h' * h)$ *are defined and*

$$\Psi(h * h') = h * \Psi(h'),$$
$$\Psi(h' * h) = \Psi(h') * h$$

.

Proof follows directly from 2.2. □

Remark 2.5. *Asymptotic semigroups.* Before going on with the argument let us explain the geometric picture standing behind it.

Recall from [V1] that for a semisimple algebraic group $\underline{G}$ there exists a canonical way to degenerate $\underline{G}$ into one of its asymptotic cones $\overline{\underline{C}}_P$ isomorphic to the affine closure of the algebraic variety $\underline{(G/U \times G/U^-)/L}$.

More precisely, in [V1] Vinberg constructs the so called *universal semigroup* $\tilde{G}$ enjoying the following properties. (He works over an algebraically closed field of characteristic 0; his construction carries over immediately to the case of a split group over a field of characteristic 0).

Let R be the set of simple roots of G; let $\overline{T} \cong (\mathbb{A}^1)^R$ be the canonical partial compactification of the abstract Cartan group of the adjoint group G/A (where A is the finite center of G). Notice that $\overline{T}$ has a natural "coordinate cross" stratification, with the set of strata being in bijection with subsets of R, i.e. with conjugacy classes of parabolics. For a parabolic P let $T_P \subset \overline{T}$ be the corresponding stratum, and let $e_P \in T_P$ be the point all of whose coordinates are either 0 or 1 (so the coordinate of e_P corresponding to a (co)root $r \in \Sigma_L$ is 0, and the one corresponding to a (co)root in $\Sigma - \Sigma_L$ is 1).

$\tilde{G}$ is an algebraic semigroup equipped with a flat morphism $\wp : \tilde{G} \to \overline{T}$, such that the preimage of the open stratum is isomorphic to $G \times_A \overline{T}$. Further, for $x \in T_P$ the preimage $\wp^{-1}(x)$ is isomorphic to the algebraic variety $\underline{(G/U \times G/U^-)/L}$.

Now the canonical bijection Ψ can be characterized as follows.

For any parabolic P there exists an A_L-invariant open neighborhood $\mathfrak{U}_P$ of $\wp^{-1}(e_P)$, and a continous A_L-equivariant projection $\pi_P : \mathfrak{U}_P \to \wp^{-1}(e_P)$, such that $\pi_P|_{\wp^{-1}(e_P)} = Id$, and the following holds.

Whenever $KgK \subset \mathfrak{U}_P$ and $KxK \subset \mathfrak{U}_P$ for $g \in G$ and $x \in C_{P'}$, where $P' \supset P$, we have $KxK = \Psi(KgK) \Leftrightarrow \pi_P(KxK) = \pi_P(KgK)$. In other words KxK and KgK are close in the topology of $\tilde{G}$.

Notice also that the domain of definition of $\Psi : K\backslash G/K \to K\backslash(C_P)/K$ is a neighborhood of the stratum corresponding to P in the DeConcini-Procesi compactification of G [DCPr].

Probably it is possible to work directly with this geometric definition; we however found it more economical (though less transparent) to use the techniques employed here.

Lemma 2.6. *Suppose that $l \in L_\mu$, where $\langle \mu, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$. Then $\Psi(KlK) = K\psi(l)K$.*

Proof Obvious. $\square$

For a partial order $\blacktriangleleft$ on $\mathfrak{a}$ we write $G_{\blacktriangleleft ?} = \bigcup\limits_{\mu \blacktriangleleft ?} G_\mu$, $Geom_{\blacktriangleleft ?}(\mathcal{H}) = \bigoplus\limits_{\mu \blacktriangleleft ?} \mathcal{H}_\mu$.

We are now ready for the proof of the next

Theorem 2.7. *For dominant enough λ we have $\mathcal{H}_{\preccurlyeq \mu} \cdot Geom_{\preccurlyeq \lambda}(\mathcal{H}) \subset Geom_{\preccurlyeq \lambda + \mu}(\mathcal{H})$.*

Lemma 2.8. *For $h \in \mathcal{H}$ and $\lambda \in \mathfrak{a}^+$ the following are equivalent:*

i) $h \in \mathcal{H}_{\preccurlyeq \lambda}$.

ii) For any parabolic, and any function $f \in C_c^\infty(G/U)^K$ we have $\operatorname{supp}(f) \subset (G/U)_\mu \Rightarrow \operatorname{supp}(h * f) \subset \bigcup\limits_{\nu \preccurlyeq \lambda + \mu} (G/U)_\nu$.

Proof Assume that ii) holds. Recall that for any $\rho \in Cusp^K$ we have $\Pi_\rho^K = C_c^\infty(G/U)^K \otimes_{\mathcal{H}(L^c)} \rho$, and the grading on Π_ρ comes from the grading on the first multiple in the RHS, given by $C_c^\infty(G/U)^K = \bigoplus\limits_\mu C_c^\infty(G/U)_\mu^K$. ii) $\Rightarrow$ i) is now obvious.

Conversely, suppose that ii) does not hold, i.e. for some $\nu \not\preccurlyeq \lambda$ there exists $f \in C_c^\infty(G/U)^K$ such that $\operatorname{supp}(h * f) \cap (G/U)_{\lambda+\nu} \neq \emptyset$. Thus the morphism of free $\mathcal{H}(L^c)$-modules $C_c^\infty(G/U)_\mu^K \to C_c^\infty(G/U)_{\mu+\nu}^K$, $f \mapsto h * f|_{(G/U)_{\mu+\nu}}$ is nonzero. Hence (recall that $\mathcal{H}(L^c)$ is finite over its center) there exists an irreducible representation ρ of L^c such that tensoring the latter morphism over $\mathcal{H}(L)$ with ρ we still obtain a nonzero morphism. Further, ρ is a subquotient in a representation of the form $i_{L_1}^L(\rho_1)|_{L^c}$, where ρ_1 is a cuspidal representation of a standard Levi $L_1 \subset L$. Then there exists $\nu_1 \in \mathfrak{a}_{L_1}$ with $\mathfrak{P}_L(\nu_1) = \nu$ such that the morphism $C_c^\infty(G/U_1)_\mu^K \to C_c^\infty(G/U)_{\mu+\nu_1}^K$, $f \mapsto h * f|_{(G/U)_{\mu+\nu_1}}$ tensored with ρ_1 is nonzero. This means that $\operatorname{supp}^{spec}(I_{\rho_1'}(h)) \ni \nu_1$, where $\rho_1' \in Cusp$ differs from ρ by twisting with an unramified character; so $\operatorname{supp}^{spec}(h) \ni \nu_1$.

If i) holds then $\nu_1 \preccurlyeq \lambda$. Since $\lambda \in \mathfrak{a}^+$ we have $\mathfrak{P}(\lambda) \preccurlyeq \lambda$; since $\mathfrak{P}_L$ preserves $\preccurlyeq$ (by [BW], Lemma 6.4 on p.139, statement 2) we get $\lambda \succcurlyeq \mathfrak{P}_L(\lambda) \succcurlyeq \nu$. We arrived at a contradiction, which proves i) $\Rightarrow$ ii). □

Lemma 2.9. *Let $\mathfrak{C} \subset G$ be a compact set. Suppose that $x_1 \in (C_P)_{\mu_1}$, $x_2 \in (C_P)_{\mu_2}$, where $\langle \mu_1, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$, and $x_2 = cx_1$ for $c \in \mathfrak{C}$. Identify the left G-orbit on C_P passing through x_1, x_2 with G/U, and suppose that $x_1 \in (G/U)_{\nu_1}$, $x_2 \in (G/U)_{\nu_2}$. Then $\nu_2 - \nu_1 = \mathfrak{P}_L(\mu_2 - \mu_1)$.*

Proof We have a map

$$W_{aff} \times_{W_{aff}(L)} W_{aff} \to K_0 \backslash C_P / K_0$$

which sends (w_1, w_2) to $K_0 \cdot (w_1, w_2) \cdot K_0$ (it is immediate to check that it is actually well-defined). For $x \in W_{aff} \times_{W_{aff}(L)} W_{aff}$ denote the corresponding double coset by $(C_P)_x$.

As in 2.3 we pick a finite set $\{w_i\} \in W_{aff}$ so that for any $w \in W_{aff}$ we have $w_i w w_j \in X^+$ for some i, j. For $\langle \mu, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$ Proposition 2.2 implies that $(C_P)_{(w_i\mu, w_j)} = \bigcup\limits_{g \in G_{w_i \mu w_j}} \Psi(KgK)$.

In particular, $x_i \in (C_P)_{w_i \lambda_i w_i'}$ for $i = 1, 2$, where λ_i is such that $\langle \lambda_i, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$, and $w_i \lambda_i w_i' \in W_f^p \cdot \mu_i \cdot W_f^p$.

Since x_1, x_2 lie in one left G-orbit we have $(\lambda_2 w_2') L_0^c \subset P^-(\lambda_1 w_1') L_0^c$, which is possible only if $w_2' \in W_{aff}(L) w_2$. Thus we can (and will) assume that $w_1' = w_2' = w'$.

Pick $x \in (C_P)_{w'} \cap G(x_1)$, and identify G/U with the left orbit $G(x_1) = G(x_2)$ by means of the map $g \mapsto g(x)$. Then we get $x_i \in (K_0 \cdot (w_i \lambda_i) \cdot (K_0 \cap L) U)/U \subset (G/U)_{\overline{w_i \lambda_i}}$, where $\overline{w}$ is the image of an element $w \in W_{aff}$ under the projection $W_{aff} \to W_f^p \backslash W_{aff} / W_{aff}^L$.

Thus $\nu_i = \mathfrak{P}_L(\tilde{\nu}_i)$, where $w_i \lambda_i \in W_f^p \cdot \tilde{\nu}_i$, while $W_f^p \mu_i W_f^p \ni w_i \lambda_i w'$. Hence $\mu_i \in W(\tilde{\nu}_i + \mu')$, where μ' is such that $w' \in \mu' \cdot W_f^p$. Notice that since $\langle \mu_i, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$, and w', μ' belong to a fixed finite set, we have $\langle \tilde{\nu}_i + \mu', \Sigma^+ - \Sigma_L^+ \rangle \gg 0$. Since μ_i is dominant, it follows that $\mu_i \in W_L(\tilde{\nu}_i + \mu')$. In particular $\mathfrak{P}_L(\mu_i) = \mathfrak{P}_L(\tilde{\nu}_i + \mu') = \nu_i + \mathfrak{P}_L(\mu')$. The Lemma follows. □

Proof of Theorem 2.7. Consider the "Rees algebra" $Rees(\mathcal{H}) := \bigoplus\limits_{\lambda \in X^+} \mathcal{H}_{\preccurlyeq \lambda}$. The explicit description of the center $Z \subset \mathcal{H}$ ([BD], Theorem 2.13) readily implies that the central subalgebra $Rees(Z) \subset Rees(\mathcal{H})$ is of finite type. Further, from Lemma 1.13b) it is not hard to deduce that there exists $\lambda_0 \in X^+$ such that for $\lambda \in \mathfrak{a}_+$, $\rho \in Cusp_L^K$ we have $h \in \mathcal{H}_{\preccurlyeq \lambda}, \nu \in \mathrm{supp}^{spec}(I_\rho(h)) \Rightarrow w(\nu) \preccurlyeq \lambda + \lambda_0$ whenever $w(L)$ is a standard Levi.

Thus we have $Rees(\mathcal{H}) \hookrightarrow \bigoplus\limits_{L, \rho \in Cusp_L^K} \bigoplus\limits_{\lambda \in \mathfrak{a}_+} \bigcap\limits_{w} \mathrm{End}(\Pi_\rho)_{w(\mathfrak{a}_{\preccurlyeq \lambda + \lambda_0})}$. The RHS is a finite $Rees(Z)$-module. Hence the LHS is, thus $Rees(\mathcal{H})$ is a Noetherian graded algebra, and in particular is finitely generated.

Let $h_i \in \mathcal{H}_{\preccurlyeq \lambda_i}$, $i = (1, .., n)$ be the generators. Then it is enough to ensure that $f \in Geom_{\preccurlyeq \mu}(\mathcal{H}) \Rightarrow h_i f \in Geom_{\preccurlyeq \lambda_i + \mu}(\mathcal{H})$.

Pick a number N which satisfies the condition of 2.4 with $\mathfrak{C} = \cup \mathrm{supp}(h_i)$ for all P.

Also let N' be such that $h_i \in Geom_{\underset{\alpha}{\leqslant} N'}(\mathcal{H})$ for all i, α.

Suppose that $\lambda \in \mathfrak{a}^+$ is large enough, $\mu \preccurlyeq \lambda$, and $h \in \mathcal{H}_\mu$.

Pick a simple root α. We consider two cases.

I. $\langle \mu, \alpha \rangle > N$. Then let P be the maximal proper parabolic corresponding to α. Applying 2.4 we see that $h_i * h = \Psi^{-1}(h_i * \Psi(h))$.

We can decompose $C_P = \bigcup_{x \in P^- \backslash G/K} (C_P)_x$, where $(C_P)_x$ is the fiber of the projection $C_P \to P^- \backslash G/K$, $(g_1 U, g_2 U^-)L \mapsto P^- g_2 K$.

Let us now write h as the sum of h_x, where $\mathrm{supp}(\Psi(h_x)) \in (C_P)_x$.

We have an isomorphism of G-modules $C_c^\infty((C_P)_x))^K \cong C_c^\infty(G/U)^{K \cap P}$. (Here the invariants are taken with respect to the right action).

Hence applying to each h_x 2.8 and 2.9 we obtain:

$$h_i * h \in Geom_{\underset{\alpha}{\leqslant} \langle \mu, \omega_\alpha \rangle + \langle \lambda_i, \omega_\alpha \rangle}(\mathcal{H}) \subset Geom_{\underset{\alpha}{\leqslant} \langle \lambda, \omega_\alpha \rangle + \langle \lambda_i, \omega_\alpha \rangle}(\mathcal{H}).$$

II. If on the other hand $\langle \mu, \alpha \rangle \leqslant N$ then

$$\langle \lambda - \mu, \omega_\alpha \rangle \geqslant \frac{(\lambda, \alpha) - N}{(\alpha, \alpha)}, \tag{34}$$

because the RHS is the distance from λ to the hyperplane $\{\eta \mid \langle \eta, \omega_\alpha \rangle = N\}$, while μ is separated from λ by this hyperplane.

If λ is large then the RHS of (34) is greater than N', so in that case

$$h_i * h \in Geom_{\underset{\alpha}{\leqslant} \langle \mu, \omega_\alpha \rangle + N'}(\mathcal{H}) \subset Geom_{\underset{\alpha}{\leqslant} \lambda}(\mathcal{H}) \subset Geom_{\underset{\alpha}{\leqslant} \lambda + \lambda_i}(\mathcal{H}).$$

Thus in any case $h_i * Geom_{\preccurlyeq \lambda}(\mathcal{H}) \subset Geom_{\preccurlyeq \lambda + \lambda_i}(\mathcal{H})$ provided λ is large.

Since h_i generate $Rees(\mathcal{H})$ this finishes the proof. □

To derive an important corollary we need some information on compatibility of our filtrations with convolution.

Lemma 2.10. *We have*

a) $Geom_{\preccurlyeq \nu_1}(\mathcal{H}) \cdot Geom_{\preccurlyeq \nu_1}(\mathcal{H}) \subseteq Geom_{\preccurlyeq \nu_1 + \nu_2}(\mathcal{H})$.

b) $G_\lambda((G/U)_\mu) \subset \bigcup_{\nu \preccurlyeq \lambda + \mu} ((G/U)_\nu)$.

Proof Recall that we have fixed a minimal Levi subgroup in good relative position with K_0. This choice determines an appartment $\mathfrak{A}$ of the Bruhat-Tits building X containing the K_0-fixed point p. Let $\Xi : X \to \mathfrak{A}$ be the contraction centered at an open polysimplex, which contains p in its closure. Then an element $g \in G$ lies in G_ν iff the coweight $\Xi(g(p)) - p \in \mathfrak{a}$ is W-conjugate to ν. Now the proof is parallel to the usual proof of the fact that Ξ does not increase distances (see [BT1] 7.4.20(ii)).

Take now $g_1 \in G_{\nu_1}$, $g_2 \in G_{\nu_2}$; we must check that $g_1 g_2 \in G_\nu$ where ν satisfies $\nu \preccurlyeq \nu_1 + \nu_2$.

Let us break the segment $[g_1(p), g_1 g_2(p))$ into the union $[z_0 = g_1(p), z_1)$, $[z_1, z_2)$, $\ldots [z_{n-1}, g_1 g_2(p) = z_n)$, where z_i and z_{i+1} lie in the closure of one open polysimplex (*loc. cit.* 7.4.21).

According to [BT1] 7.4.19, 7.4.18(i) the map Ξ restricted to the closure of any polysimplex coincides with the action of some $g \in G$, hence we have $\Xi(z_i) = x_i(z_i)$, $\Xi(z_{i+1}) = x_i(z_{i+1})$ for some $x_i \in G$.

On the other hand $\Xi([p, g_2(p)]) = g([p, g_2(p)])$ for some $g \in G$ is a line segment, and by the above remark $w(\Xi(g_2(p)) - p) = \nu_2$ for some $w \in W$.

Then obviously $\nu_2 = \sum_{i=0}^{n-1} \nu^i$, where $\nu^i = w\left(\Xi(g_2(z_i)) - \Xi(g_2(z_{i-1}))\right) \in \mathfrak{a}^+$.

By [BT1] 7.4.8 for points $p_1, p_2, p_1', p_2' \in \mathfrak{A}$ the elements $p_1 - p_2$ and $p_1' - p_2' \in \mathfrak{a}$ are W-conjugate provided $p_i = g(p_i')$ for some $g \in G$.

Thus we see that $\Xi(z_i) - \Xi(z_{i-1})$ is W-conjugate to $\Xi(g_2(z_i)) - \Xi(g_2(z_{i-1})) = gg_1x_i^{-1}(\Xi(z_i)) - gg_1x_i^{-1}(\Xi(z_{i-1}))$, and hence also to ν^i. Let $w_i \in W$ be such that $\Xi(z_i) - \Xi(z_{i-1}) = w_i(\nu^i)$. Suppose that $g_1g_2 \in G_\nu$, i.e. $w'(\Xi(g_1g_2(p) - p) = \nu$ for some $w' \in W$. Then we have

$$\nu = w'(\Xi(g_1g_2(p) - p) = w'(\nu_1 + \sum w_i(\nu^i)) = w'(\nu_1) + \sum w'w_i(\nu^i) \preccurlyeq \nu + \sum \nu^i = \nu_1 + \nu_2.$$

This proves a).

To prove b) we reinterpret the condition $gU \in (G/U)_\mu$ as follows. Let $\Xi' : X \to \mathfrak{A}$ be the contraction centered at a "vector chamber" corresponding to P_0. Then $gU \in (G/U)_\mu$ if and only if $\mathfrak{P}_L(p - \Xi'(g^{-1}(p))) = \nu$.

From the definition ([BT1] 7.4.25) it is clear that $\Xi'|_{\overline{\Delta}} = g_\Delta$ for any polysimplex for some $g_\Delta \in G$. Hence repeating the arguments used in a) we can find $\lambda^i \in \mathfrak{a}^+$ and $w_i \in W$, such that $\Xi(g_2^{-1}(p)) - \Xi(g_2^{-1}g_1^{-1}) = \sum w_i(\lambda^i)$, while $\lambda = \sum \lambda^i$. Since by [BW], Lemma 6.4 on p.139 $\mathfrak{P}_L$ preseves $\preccurlyeq$ we see that if $g_1 \in G_\lambda$, $g_2U \in (G/U)_\mu$, $g_1g_2 \in (G/U)_{\mu'}$ then

$$\mu' = \mathfrak{P}_L(p - \Xi'(g_2^{-1}(p))) + \mathfrak{P}_L(\Xi(g_2^{-1}(p)) - \Xi(g_2^{-1}g_1^{-1})) = \mathfrak{P}_L(\mu) + \mathfrak{P}_L(\sum w_i(\lambda^i)) \preccurlyeq \mu + \sum \lambda_i = \mu + \lambda.$$

This proves the Lemma. □

Corollary 2.11. *For any $\lambda \in X^+$ we have $\mathcal{H}_{\preccurlyeq\lambda} \supset Geom_{\preccurlyeq\lambda}(\mathcal{H})$.*

Proof Compare Lemma 2.8 with Lemma 2.10b). □

The latter statement together with Theorem 2.7 yield the following

Proposition 2.12. *Fix λ_0 in X^+, and define filtration $F_{\leqslant\lambda}(\mathcal{H}) := \sum_{\mu\in\mathfrak{a}^+_{\leqslant\lambda}} Geom_{\preccurlyeq\lambda_0+\mu}(\mathcal{H})$. Then $F_\leqslant$ is a good filtration on the free $\mathcal{H}$-module, provided λ_0 is large enough.*

Proof From Theorem 2.7 we see that the filtrations on the algebra and on the module are compatible if λ_0 is large.

It remains to show that for all but finitely many ν we have

$$F_{\leqslant\lambda} \subset \sum_{\lambda_2\in\mathfrak{a}^+_{<\lambda}} \sum_{\lambda_1\in\mathfrak{a}_{\leqslant\lambda-\lambda_2}} \mathcal{H}_{\leqslant\lambda_1} \cdot F_{\leqslant\lambda_2}. \tag{35}$$

We will in fact show that

$$F_{\leqslant\lambda} \subset \sum_{\lambda_2\in\mathfrak{a}^+_{<\lambda}} \sum_{\lambda_1\in\mathfrak{a}_{\leqslant\lambda-\lambda_2}} Geom_{\preccurlyeq\lambda_1}(\mathcal{H}) \cdot F_{\leqslant\lambda_2} \tag{36}$$

for almost all λ; (36) is stronger than (35) by 2.11.

Pick some λ and ν such that $\mathcal{H}_\nu \subset F_{\leqslant\lambda}(\mathcal{H})$. We want to check that $\mathcal{H}_\nu$ lies in the RHS of (36).

By the definition $\nu \preccurlyeq \lambda' + \lambda_0$ for some $\lambda' \in \mathfrak{a}^+_{\leqslant\lambda}$. If $\lambda' < \lambda$ then $\mathcal{H}_\nu$ does lie in the RHS of (36), for it lies in the summand corresponding to $\lambda_2 = \lambda'$, $\lambda_1 = 0$. So assume that $\lambda' = \lambda \succcurlyeq \nu$.

For any $w \in W^p_f$ consider the intersection $w^{-1}W_{aff}w \cap \Lambda$. It is a sublattice of finite index in Λ; in particular there exists some $n \in \mathbb{Z}^{>0}$ such that $n\omega_\alpha \in w^{-1}W_{aff}w \cap \Lambda$ for any $w \in W^p_f$ and any simple coroot α.

For all but finitely many λ there exists a simple coroot α such that $(\lambda, \omega_\alpha) \geqslant n+1$; thus we can suppose that λ, α are like that. If $\nu \underset{\alpha}{\leqslant} n$ then $\mathcal{H}_\nu \subset F_{\leqslant\lambda-\omega_\alpha}$, so we can take $\lambda_1 = 0$, $\lambda_2 = \lambda - \omega_\alpha$.

It remains to consider the case $(\nu, \omega_\alpha) > n$.

We have $\mathcal{H}_\nu = \oplus\mathcal{H}_x$, where x runs over double cosets $W_pwW_p \in W_p\backslash W_{aff}/W_p = K_0\backslash G/K_0$ such that $w \in W_{aff} \cap (W^p_f \nu W^p_f)$.

Fix one such w; let us write it down as $w = w_1\nu w_2$, $w_i \in W^p_f$, and consider the element $y = W_p \cdot (w_1(n\omega_i)w_1^{-1}) \cdot W_p \in W_p\backslash W_{aff}/W_p$.

It is obvious that for any three double cosets z_1, z_2, z_3 where $z_1 = W_p \cdot w'W_p$, $z_2 = W_p \cdot w''W_p$, $z_3 = W_p \cdot w'w''W_p \in W_p\backslash W_{aff}/W_p$ we have $\mathcal{H}_{z_1} \cdot \mathcal{H}_{z_2} \supset \mathcal{H}_{z_3}$.

Hence we have $\mathcal{H}_x \subset \mathcal{H}_y \cdot \mathcal{H}_z$, where $z = W^p_f w_1(\nu - n\omega_\alpha)w_2$.

Since $\mathcal{H}_z \subset \mathcal{H}_{\nu-n\omega_\alpha} \subset \mathcal{H}_{\preccurlyeq(\lambda-n\omega_\alpha)+\lambda_0} \subset F_{\leqslant\lambda-n\omega_\alpha}$ we see that in this case $\mathcal{H}_\nu$ lies in the summand of the RHS of (36) corresponding to $\lambda_1 = n\omega_i$, $\lambda_2 = \lambda - n\omega_i$. The Proposition is proved. $\square$

Remark 2.13. We will also use the following equivalent description of $F_{\leqslant}$.

For a coweight $\lambda \in \mathfrak{a}$ let $\wp(\lambda)$ denote the point of $\mathfrak{a}_+$ closest to λ. Then

$$\begin{aligned} \wp(\lambda) &= \sum_\alpha a_\alpha\omega_\alpha \\ \wp(\lambda) - \lambda &= \sum_\beta b_\beta\beta; \end{aligned} \tag{37}$$

where α and β run over nonintersecting sets of simple roots, and $a_\alpha > 0$, $b_\beta \geqslant 0$; moreover for any coweight λ there exists a unique such decomposition (the latter statement is the so-called Langlands combinatorial Lemma, see e.g. [BW], Lemma 6.11 on p.143).

It is not hard to see that $\wp(\mu)$ is the minimal element in the set $\{\lambda \in \mathfrak{a}_+ \mid \lambda \succcurlyeq \mu\}$. It follows that

$$F_{\leqslant\lambda}(\mathcal{H}) = \bigoplus_{\wp(\mu-\lambda_0)\leqslant\lambda} \mathcal{H}_\mu \tag{38}$$

2.14. Another result we will need is the following

Theorem 2.15. *Fix $\lambda_1 \in \mathfrak{a}^+$. Make the following assumptions:*

$(e_1, r_j) > 0$ for all $r_j \in \Sigma^+$;

h^λ_x (notation of 1.17) satisfies the condition of 1.17 with large enough a (the bound on a depends on λ_1);

$\mu \in \mathfrak{a}_+$ is very dominant.

Then we have:

a) If $\nu \in \mathfrak{a}_+$ is such that $\langle\nu, \Sigma^+ - \Sigma^+_L\rangle \gg 0$ then

$$\wp(\nu - \lambda_0) \leqslant \mu + \lambda_1,\ h \in \mathcal{H}_\nu \Longrightarrow \Psi(h^\lambda_x h) = \iota(\lambda)m_x\Psi(h) + \Psi(h'),$$

where $h' \in F_{\leqslant\lambda+\mu}(\mathcal{H})$.

b) Otherwise $h^\lambda_x h \in F_{\leqslant\lambda+\mu}(\mathcal{H})$.

Lemma 2.16. *For $L' \subset L$ and $f \in C_c^\infty(G/U')_\mu^K$ we have $h_x^\lambda f = \iota(\lambda) m_x(f) + f'$, where $f' \in \sum\limits_{r \in \Sigma^+ - \Sigma_L^+} C_c^\infty(G/U')_{\preccurlyeq \mu+\lambda-a\cdot\mathfrak{P}_{L'}(r)}$.*

Proof is similar to the proof of Lemma 2.8. It is enough to prove that for any irreducible representation ρ_L of L the action of $h_x^\lambda - \iota(\lambda)$ on $\Pi_{\rho_L}^K = C_c^\infty(G/U)^K \otimes_{\mathcal{H}(L^c)} \rho_L$ belongs to $\sum\limits_{\alpha \in \Sigma^+ - \Sigma_L^+} \mathrm{End}(\Pi_{\rho_L})_{\preccurlyeq \lambda - ar}$. The irreducible representation ρ_L is a subquotient in the induced representation $i_{L'}^L(\rho_{L'})$, where $\rho_{L'}$ is a cuspidal representation of a standard Levi. Then $\Pi_{\rho_L} = \Pi_{\rho_{L'}}$, and the statement reduces to $I_{\rho_{L'}}(h_x^\lambda) - [\lambda]_{\rho_{L'}} \in \sum\limits_{\alpha \in \Sigma^+ - \Sigma_L^+} \mathrm{End}(\Pi_{\rho_{L'}}^K)_{\preccurlyeq \lambda - ar}$, which follows from Proposition 1.7. □

Lemma 2.17. *For a weight $\nu \in \mathfrak{a}_+$ such that $\langle \nu, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$ we have $\wp(\nu + \lambda_1 - a \cdot r - \lambda_0) \leqslant \wp(\nu - \lambda_0)$ for $r \in \Sigma^+ - \Sigma_L^+$, provided $a > \frac{(e_1,\lambda_1)}{(e_1,\alpha)}$ for any simple root $\alpha \in \Sigma^+ - \Sigma_L^+$.*

Proof Let $\alpha \in \Sigma^+ - \Sigma_L^+$ be a simple root such $\alpha \preccurlyeq r$.

If ν satisfies $\langle \nu, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$ then also $\wp(\nu - \lambda_0) + \lambda_1 - a\alpha, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$. Since $\langle \alpha, \beta \rangle \leqslant 0$ for $\beta \in \Sigma_L^+$ it follows that $\wp(\nu - \lambda_0) + \lambda_1 - a\alpha \in \mathfrak{a}_+$. From the condition of Lemma we get $(e_1, \wp(\nu - \lambda_0) + \lambda_1 - a\alpha) < (e_1, \wp(\nu - \lambda_0))$, hence $\wp(\nu - \lambda_0) + \lambda_1 - a\alpha \leqslant \wp(\nu - \lambda_0)$. The Lemma now follows from (38). □

Proof of Theorem 2.15 follows the same scheme as the proof of 2.7.

Let L' be minimal among the standard Levi subgroups for which $\Psi_{L'}|_{G_\nu}$ is defined. Under the conditions of a) we have $L' \subset L$.

In the notations of the proof of 2.7, we can assume that $\Psi(h) \in \mathcal{H}(C_{P'})_x$ for some $x \in P^- \backslash G/K$. Then using Lemma 2.16 together with 2.8, 2.9 we get

$$\Psi(h_x^\lambda h) - m_x \iota(\lambda) \Psi(h) \in \sum_{r \in \Sigma^+ - \Sigma_L^+} \sum_{\eta \preccurlyeq \nu + \lambda - ar} \mathcal{H}(C_{P'})_\eta,$$

hence

$$\mathrm{supp}^{geom}(h_x^\lambda h - \Psi^{-1}(m_x \iota(\lambda) \Psi(h))) \subset \bigcup_{r \in \Sigma^+ - \Sigma_L^+} \bigcup_{\mathfrak{P}_{L'}(\eta) \preccurlyeq \nu + \lambda - ar} G_\eta.$$

Let, first, η be such that $G_\eta \cap \mathrm{supp}^{geom}(h_x^\lambda h - \Psi^{-1}(m_x \iota(\lambda)\Psi(h))) \neq \emptyset$. As we have just seen there exists $r \in \Sigma^+ - \Sigma_L^+$ such that $\eta \underset{\alpha}{\leqslant} \nu + \lambda - ar$ for any simple coroot $\alpha \in \Sigma^+ - \Sigma_{L'}^+$. By Lemma 2.17 $\nu + \lambda - ar \preccurlyeq \lambda_0 + \lambda'$, where $\lambda' \leqslant \mu + \lambda$.

Thus for a simple coroot $\alpha \in \Sigma^+ - \Sigma_{L'}^+$ we have $\eta \underset{\alpha}{\leqslant} \nu + \lambda - ar \underset{\alpha}{\leqslant} \lambda_0 + \lambda'$.

On the other hand, if α is a simple coroot of L' then $\langle \omega_\alpha, \nu \rangle$ is bounded. Since λ_0 is very dominant we have $\eta \underset{\alpha}{\leqslant} \lambda_0 \underset{\alpha}{\leqslant} \lambda_0 + \lambda'$.

So $\eta \underset{\alpha}{\leqslant} \lambda_0 + \lambda'$ for any simple coroot α, i.e. $\eta \preccurlyeq \lambda_0 + \lambda'$, which implies $h_x^\lambda h - \Psi^{-1}(m_x \iota(\lambda) \Psi(h)) \in F_{\leqslant \mu + \lambda}(\mathcal{H})$. a) is proved.

Assume now we are in the situation of b). Then for some simple root $\alpha \in \Sigma_L^+$ the number $\langle \nu, \omega_\alpha \rangle$ is bounded.

The condition $(e_1, \alpha) > 0$ implies that we can find $a \in \mathbb{R}_{>0}$ such that $\lambda_1 - a\alpha \leqslant 0$; we can then assume $\lambda \succcurlyeq a\alpha$. Also we can assume (after possibly increasing λ_0) that $\mathcal{H}_{\preccurlyeq \lambda} \cdot Geom_{\preccurlyeq \mu}(\mathcal{H}) \subset Geom_{\preccurlyeq \lambda + \mu}$ holds already for $\mu \succcurlyeq \lambda_0 - a\alpha$. By the

definition $\nu \preccurlyeq \lambda_0+\zeta$ for some $\zeta \leqslant \mu+\lambda_1$. But since $\langle \omega_\alpha, \nu\rangle$ is bounded we also have $\nu \preccurlyeq \lambda_0+\zeta-a\alpha$. Then we have $h_x^\lambda h \in Geom_{\preccurlyeq \lambda_0+\zeta-a\alpha+\lambda} \subset F_{\leqslant \lambda-a\alpha+\zeta} = F_{\leqslant \lambda+\mu+(\lambda_1-a\alpha)} \subset F_{\leqslant \lambda+\mu}$. $\square$

3. End of the proof: orbital integrals.

Theorem 3.1. *The functional $Tr(g,M)$ defined above satisfies 0.23iii).*

We start with recalling the following standard computation.

We will say that a subset $\mathfrak{S} \subset \Lambda^+$ is large if it contains $(\Lambda^+)_{\preccurlyeq\lambda}$ for large λ.

For any $\mathfrak{S} \subset \Lambda^+$ denote $G_{\mathfrak{S}} = \bigcup_{\lambda\in\mathfrak{S}} G_\lambda$, $\mathcal{H}_{\mathfrak{S}} = \bigoplus\limits_{\lambda\in\mathfrak{S}} \mathcal{H}_\lambda$ and let $pr_{\mathfrak{S}} : \mathcal{H}^{\oplus m} \to \mathcal{H}_{\mathfrak{S}}^{\oplus m} \subset \mathcal{H}^{\oplus m}$ be the projector along $\mathcal{H}^{\oplus m}_{\Lambda^+-\mathfrak{S}}$.

Lemma 3.2. *Keep the assumptions of 3.1. Then for any large enough finite set $\mathfrak{S} \subset \Lambda^+$ we have*

$$O_{g^{-1}}(\sum E_{ii}) = tr(g\circ E\circ pr_{\mathfrak{S}})$$

(the RHS is the well-defined trace of a finite-rank endomorphism of $\mathcal{H}^{\oplus m}$.)

Proof of the Lemma. We have $tr(g\circ E\circ pr_{\mathfrak{S}}, \mathcal{H}^{\oplus m}) = tr(\delta_{g\cdot K}\circ E\circ pr_{\mathfrak{S}}, k[G/K]^{\oplus m}) = \int_{g'\in K\cdot g} tr(g'\circ E\circ pr_{\mathfrak{S}}^{\oplus m}, k[G/K]^{\oplus m})$, where we reused the symbol $pr_{\mathfrak{S}}$ to denote also the projector $k[G/K]^{\oplus m} \to k[G_{\mathfrak{S}}/K]^{\oplus m} \subset k[G/K]^{\oplus m}$ along $k[G_{\Lambda^+-\mathfrak{S}}/K]^{\oplus m}$.

In $k[G/K]$ we have the standard basis consisting of delta-functions; it gives also a basis in $k[G/K]^{\oplus m}$. We use this basis to "compute" the trace of $g'\circ E\circ pr_{\mathfrak{S}}^{\oplus m}$. Precisely, let dg be the Haar measure on G such that $\int_K dg = 1$; if $x \in G/K$, and $x_{(i)} = (\underbrace{0,\ldots,0}_{i-1},\delta_x,\underbrace{0,\ldots,0}_{m-i})$ is a basis element of $\mathcal{H}^{\oplus m}$, then the corresponding diagonal entry of $g'\circ E\circ pr_{\mathfrak{S}}^{\oplus m}$ is equal to

$$\begin{cases} 0 \quad \text{if} \quad x\notin G_{\mathfrak{S}}/K \\ \frac{E_{ii}}{dg}\left(x^{-1}(g')^{-1}x\right) \quad \text{if} \quad x\in G_{\mathfrak{S}}/K \end{cases} \tag{39}$$

If $K\cdot g \ni g'$ is contained in the regular elliptic set, then for $\mathfrak{S}$ large enough we have $\frac{E_{ii}}{dg}(x^{-1}(g')^{-1}x) \neq 0 \Rightarrow x \in G_{\mathfrak{S}}/K$. For such $\mathfrak{S}$ summation of (39) over x gives $O_{(g')^{-1}}(E_{ii})$.

Altogether we get

$$tr(g\circ E\circ pr_{\mathfrak{S}}, \mathcal{H}^{\oplus m}) = tr(\int_{g'\in K\cdot g} g'\circ E\circ pr_{\mathfrak{S}}, k[G/K]^{\oplus m})dg' =$$

$$= \int_{g'\in K\cdot g} O_{(g')^{-1}}(\sum E_{ii}) = O_{g^{-1}}(\sum E_{ii})$$

where the last equality follows from the assumption that $O_{(g')^{-1}}(\sum E_{ii})$ is constant for $g' \in g\cdot K$. $\square$

We proceed now to the proof of the Theorem.

Fix very dominant λ_0, and set $\mathfrak{S}_\lambda = \bigcup\limits_{\mu\leqslant\lambda} X^+_{\preccurlyeq\mu+\lambda_0}$. Then $\mathcal{H}^{\oplus m}_{\mathfrak{S}_\lambda} = F_{\leqslant\lambda}(\mathcal{H})^{\oplus m}$ is a good filtration on the free module of rank m by 2.12.

Thus the image of $F_{\leqslant\lambda}(\mathcal{H})^{\oplus m}$ under E is a good filtration on M, hence we have $Tr(g,M) = tr\,(g, E(F_{\leqslant\lambda}(\mathcal{H})^{\oplus m}))$ for large λ.

So to prove the Theorem we need only to show that

$$tr(g, E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m})) = tr(g \circ E \circ pr_{\mathfrak{S}_\lambda}, \mathcal{H}^{\oplus m}). \tag{40}$$

It is convenient to consider the vector space $E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}$ which is invariant under the action of E, $pr_{\mathfrak{S}_\lambda}$, g. Obviously

$$tr(g, E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m})) = tr(g \circ E, E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}),$$

$$tr(g \circ E \circ pr_{\mathfrak{S}_\lambda}, \mathcal{H}^{\oplus m}) = tr(g \circ E \circ pr_{\mathfrak{S}_\lambda}, E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}).$$

Thus the difference of the two sides of (40) coincides with

$$tr(g \circ E \circ (Id - pr_{\mathfrak{S}_\lambda}), E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) = tr(g \circ E \circ pr_{X_+ - \mathfrak{S}_\lambda}, E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) =$$
$$tr(g \circ pr_{X_+ - \mathfrak{S}_\lambda} \circ E, \left(E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}\right) \cap \mathcal{H}_{X^+ - \mathfrak{S}_\lambda}^{\oplus m}),$$

so the Theorem is reduced to the following assertion.

Proposition 3.3. *The endomorphism $g \circ pr_{X^+ - \mathfrak{S}_\lambda} \circ E$ of the vector space $\left(E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}\right) \cap \mathcal{H}_{X^+ - \mathfrak{S}_\lambda}^{\oplus m}$ is traceless for large λ.*

Denote

$$\left(E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}\right) \cap \mathcal{H}_{X^+ - \mathfrak{S}_\lambda}^{\oplus m} = pr_{X^+ - \mathfrak{S}_\lambda}\left(E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m})\right) \xrightarrow{\sim} \left(E(\mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}\right) / \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}$$

by W_λ. For $\nu \leqslant \lambda$ let us denote by $(W_\lambda)_{\leqslant \nu} \subset W_\lambda$ the subspace $\left(E\left(F_{\leqslant \nu}(\mathcal{H}^{\oplus m})\right) + F_{\leqslant \lambda}(\mathcal{H}^{\oplus m})\right) / F_{\leqslant \lambda}(\mathcal{H}^{\oplus m}) = \left(E\left(F_{\leqslant \nu}(\mathcal{H}^{\oplus m})\right) + \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}\right) / \mathcal{H}_{\mathfrak{S}_\lambda}^{\oplus m}$.

For a number N let λ_N be the maximal element in the (finite) set $X_N := \{\lambda \in X^+ \mid (\lambda, e_1) \leqslant N\}$. Then $X^+_{\leqslant \lambda_N} = X_N$; since X_N is arbitrary large for $N \gg 0$ it is enough (in view of 3.2) to prove the Proposition for $\lambda = \lambda_N$. We write $W_N, \mathfrak{S}_N$ instead of $W_{\lambda_N}, \mathfrak{S}_{\lambda_N}$.

Consider the associated graded space $grW := \bigoplus_N \bigoplus_{\lambda \in X_N} (W_N)_{\leqslant \lambda} / (W_N)_{< \lambda}$. It carries a natural action of $gr\mathcal{H}$ provided by the obvious maps $gr\mathcal{H}_\mu \otimes (W_N)_{\leqslant \lambda}/(W_N)_{<\lambda} \to (W_{N+(e_1,\mu)})_{\leqslant \mu+\lambda}/(W_{N+(e_1,\mu)})_{<\mu+\lambda}$ which are induced by multiplication in $\mathcal{H}$.

Lemma 3.4. *grW is a finitely generated $gr\mathcal{H}$-module.*

Proof It is obvious that $gr(W_N)_\lambda \neq 0 \Rightarrow N - n \leqslant \langle \lambda, e_1 \rangle \leqslant N$ where n is some constant (depending on $\mathrm{supp}^{geom}(E_{ij})$). Thus grW is a finite sum of modules $grW^{(i)} := \bigoplus_{N - (\lambda, e_1) = i} gr(W_N)_\lambda$, each one of which is a quotient of $gr_F(\mathcal{H}^{\oplus m})$, so is finitely generated. □

Now by the proof of 1.19b) we can break Λ_0^+ into a finite union of cosets

$$\Lambda_0^+ = \bigcup \mu_i + X_{L_i}^+ \tag{41}$$

so that $\lambda \in \mu_i + X_{L_i}^+ \Rightarrow grW_\lambda = M_i$ where M_i is a finite $\mathcal{A}_{L_i}$-module (notations of (31)).

Notice that for $\lambda \in \mathfrak{a}_+$ the set $\mathfrak{S}_\lambda - \mathfrak{S}_{<\lambda}$ looks as follows. Suppose that λ lies strictly inside $\mathfrak{a}_L^+$. Then $\mathfrak{S}_\lambda - \mathfrak{S}_{<\lambda} = (\lambda_0 + \wp^{-1}(\lambda - \lambda_0)) \cap \Lambda^+ = \Lambda^+ \cap \{\lambda - \sum a_i r_i \mid 0 \leqslant a_i \leqslant \langle \omega_i, \lambda_0 \rangle\}$ where r_i runs over the set of simple roots of L.

In particular we have $\langle \lambda, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$, $\nu \in \mathfrak{S}_\lambda - \mathfrak{S}_{<\lambda} \Rightarrow \langle \nu, \Sigma^+ - \Sigma_L^+ \rangle \gg 0$.

Proceeding by induction in $\dim(\mathfrak{a}_L)$ we can choose decomposition (41) so that $\langle \nu, \Sigma^+ - \Sigma_{L_i}^+ \rangle$ is arbitrary large if $\nu \in \mathfrak{S}_\lambda - \mathfrak{S}_{<\lambda}$ where $\lambda \in \mu_i + X_{L_i}^+$. In particular

we can guarantee that $G_{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}}$ lies in the domain of $\Psi : K\backslash G/K \to K\backslash (C_{P_i})/K$ for $\eta \in \mu_i + X^+_{P_i}$. We shall make this assumption from now on.

Then we get a direct sum decomposition:

$$\mathcal{H}^{\oplus m}_{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}} = \bigoplus_{x\in K\backslash G/P} \Psi^{-1} C^\infty (C_P^{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}})^{\oplus m}_x \tag{42}$$

So take $\eta \in \mu_i + X^+_{P_i}$, and consider the surjection $\pi : \mathcal{H}^{\oplus m}_{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}} \to gr(W_N)_\eta \cong E(\mathcal{H}^{\oplus m}_{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}})/\left(E(\mathcal{H}^{\oplus m}_{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}}) \cap (E(\mathcal{H}^{\oplus m}_{\mathfrak{S}_{<\eta}}) + \mathcal{H}^{\oplus m}_{\mathfrak{S}_N})\right)$.

Since its target is an $\mathcal{A}_{L_i}$-module, it also decomposes as a direct sum: $gr(W_N)_\eta = \bigoplus_{x\in K\backslash G/P} gr(W_N)^x_\eta$. It is not hard to see that π commutes with the $\mathcal{A}_{L_i}$-action, and hence sends respective summands to respective summands (i.e. $\pi\Psi^{-1}\left(C^\infty(C_P^{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}})_x\right) \subset gr(W_N)^x_\eta$).

Hence for each N, η we can choose a subspace $(V_N)_\eta \subset \mathcal{H}^{\oplus m}_{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}}$ in such a way that:

a) π maps $(V_N)_\eta$ to $gr(W_N)_\eta$ isomorphically.

b) $(V_N)_\eta = \bigoplus_{x\in K\backslash G/P} \left(\Psi^{-1}C^\infty(C_P^{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}})^{\oplus m}_x\right) \cap (V_N)_\eta$.

c) $(V_N)_\eta$ is K_0 invariant.

d) For $\mu \in X^+_{L_i}$ the isomorphism $\Psi^{-1}_{P_i} \circ \iota(\mu) \circ \Psi_{P_i} : \mathcal{H}^{\oplus m}_{\mathfrak{S}_\eta - \mathfrak{S}_{<\eta}} \cong \mathcal{H}^{\oplus m}_{\mathfrak{S}_{\mu+\eta} - \mathfrak{S}_{<\mu+\eta}}$ sends $(V_N)_\eta$ to $(V_{(e_1,\mu)+N})_{\mu+\eta}$.

Set now $(W_N)_\eta = E\left((V_N)_\eta\right)$. By property a) we have a direct sum decomposition $W_N = \bigoplus_\nu (W_N)_\nu$.

Let $E^\nu_N \in End\left((W_N)_\nu\right)$ denote the corresponding diagonal block of $pr_{X^+ - \mathfrak{S}_N} \circ E$.

To finish the proof we need the following (key)

Lemma 3.5. *For N large and any $x \in P\backslash G/K$ we have:*

a) If $\langle \nu, \Sigma^+ - \Sigma^+_L\rangle \gg 0$ then $h_{x,\lambda} \cdot (W_N)_\nu \subset (W_{N+(e_1,\lambda)})_{\nu+\lambda} + \mathcal{H}_{\mathfrak{S}_{N+(e_1,\lambda)}}$.

b) $h_{x,\lambda} \cdot (W_N)_\nu \subset \mathcal{H}_{\mathfrak{S}_{N+(e_1,\lambda)}}$ otherwise.

Proof By 2.4 property d) implies that $\Psi_P(W_N)_{\nu+\lambda} = \iota(\lambda)\Psi_P(W_N)_\nu$ if $\langle \nu, \Sigma^+ - \Sigma^+_L\rangle \gg 0$. Also from the definition it is clear that under the same condition on ν the space $\Psi((V_N)_\nu)$, and hence $\Psi((W_N)_\nu)$ is invariant under the action of m_x. So a) follows from Theorem 2.15a) (take λ_1 to be any coweight satisfying $E_{ij} \in \mathcal{H}_{\leqslant\lambda_1}$).

Statement b) follows Theorem 2.15b).

Corollary 3.6. *E^ν_N commutes with the action of $\mathcal{A}_L$ on $(W_N)_\nu \widetilde{\longrightarrow} gr(W_N)_\nu$ provided $\langle \nu, \Sigma^+ - \Sigma^+_L\rangle \gg 0$.*

Proof Fix $x \in K\backslash G/P$ and consider the element $h^\lambda_x \in \mathcal{H}$ (see 1.17), where $\lambda \in X^+_L$ is dominant enough as in 1.17.

We have $h' \in (W_N)_\nu \Rightarrow h' \cdot E = E^\nu_N(h') + \sum_{\nu'\neq\nu} h'_{\nu'} + h_0$ where $h'_{\nu'} \in (W_N)_{\nu'}$ and $h_0 \in \mathcal{H}_{\mathfrak{S}_N}$. Multiplying the last equality by h^λ_x on the left and applying 3.5 we get the statement. □

Proof of 3.3. From 3.6 it follows that for large N and any ν the endomorphism E^ν_N preserves the direct sum decomposition $(W_N)_\nu = \bigoplus_{x\in K\backslash G/P} ((W_N)_\nu)_x$. Since

g permutes these summands without fixed points, the endomorphism $g \circ E_N^\nu$ is traceless. Hence $tr(g \circ pr_{X^+-\mathfrak{S}_N} \circ E) = \sum_\nu tr(g \circ E_N^\nu) = 0$.

Proposition 3.3, and thus also Theorem 3.1 is proved. □

We can now summarize our results in the following

Theorem 3.7. *Let $\mathfrak{M}$ be a projective G-module, and let $K \subset K_0$ be a normal open subgroup, so small that:*

α) g is K-elliptic, so $Tr(g, \mathfrak{M}^K)$ is defined (see 0.22, 0.23).

β) $\mathfrak{M}$ is generated by its K-fixed vectors.

γ) Conditions of 0.23iii) are satisfied for $M = \mathfrak{M}^K$.

Then

$$Tr(g, \mathfrak{M}^K) = O_{g^{-1}}\left(\langle \mathfrak{M} \rangle\right). \tag{43}$$

Proof There exists an idempotent $E \in Mat_m(\mathcal{H})$, such that $\mathfrak{M}$ is isomorphic to the image of E acting on the right on $k[G/K]^{\oplus m}$. Since K is nice by the assumption (see 0.23iii), we have $\langle \mathfrak{M} \rangle = \sum_i E_{ii} \mod [\mathcal{H}(G), \mathcal{H}(G)]$, so the statement follows from Theorem 3.1. □

3.8. *Proof of Theorem 0.19.* Let $0 \to P_n \to \cdots \to P_0 \to \rho \to 0$ be a projective resolution of ρ. Pick a nice open compact K, such that g is K-elliptic, properties β) and γ) of 3.7 hold for $\mathfrak{M} = P_i$, and the character χ_ρ is constant on the coset $g \cdot K$. Then by additivity of $Tr(g, M)$ (0.23(i)=1.24(b)) and 3.7 we get $Tr(g, \rho^K) = O_{g^{-1}}(\langle \rho \rangle)$. On the other hand, by 0.23(ii)=1.24(c) we have $Tr(g, \rho^K) = \chi_\rho(g)$. The proof is finished. □

4. **Elliptic pairing.** It remains to deduce Theorem 0.20 from 0.19. This argument can also be found in [Sch-St2] and is included here for the sake of completeness.

We set $k = \mathbb{C}$, $\mathrm{char}(F) = 0$ till the end of the argument.

Recall that the measure $d\mu$ on the set Ell of regular semisimple elliptic conjugacy classes is characterized by the equality

$$\int_{Ell} O_g(h) d\mu(g) = \int_G h \tag{44}$$

being true for all $h \in \mathcal{H}$ supported inside the set of regular elliptic elements.

Lemma 4.1. *For any $\mathfrak{M} \in \mathcal{M}$ and $\rho \in \mathcal{R}$ we have*

$$\sum (-1)^i \dim Ext^i(\mathfrak{M}, \rho) = \int_G \chi_\rho \cdot \langle \mathfrak{M} \rangle.$$

Proof Since both sides are additive on short exact sequences in $\mathfrak{M}$, it is enough to consider the case of projective $\mathfrak{M}$. As usual we write $\mathfrak{M}$ as the image of the right action of an idempotent $E \in Mat_m(\mathcal{H})$ on $\mathcal{H}^{\oplus m}$. We have

$$\mathrm{Hom}_G(\mathcal{H}, \rho) \cong \hat{\rho}, \tag{45}$$

where $\hat{\rho} \overset{\mathrm{def}}{=} \varprojlim_K (\rho_K)$. (For $\rho \in \mathcal{R}$ we have also $\hat{\rho} = *(\check{\rho})$). The right action of $Mat_n(\mathcal{H})$ on $\mathcal{H}^{\oplus n}$ induces an action of $Mat_n(\mathcal{H})$ on $\mathrm{Hom}_G(\mathcal{H}^{\oplus n}, \rho)$, $m(f)(x) :=$

$f(m(x))$. Evidently, this action agrees under (45) with the action of $Mat_n(\mathcal{H})$ on $\hat{\rho}^{\oplus n}$ inherited from the action of $\mathcal{H}$ on ρ. Thus we have

$$\dim(\mathrm{Hom}_G(\mathfrak{M},\rho)) = \dim(\mathrm{Hom}_G(\mathrm{Im}(E,\mathcal{H}^{\oplus n}),\rho)) = \dim(\mathrm{Im}(E,\hat{\rho}^{\oplus n})) = \dim(\mathrm{Im}(E,\rho^{\oplus n})) = tr(E,\rho^{\oplus n}) = \int_G \chi_\rho \cdot \sum_i E_{ii}. \quad (46)$$

The Lemma is proved.

Lemma 4.2. *Let Φ be an invariant generalized function on G supported on the set of regular non-elliptic elements. Then for any admissible representation ρ we have:*

$$\int_G \langle\rho\rangle \cdot \Phi = 0.$$

Proof By Theorem 0 in [K] it is enough to show that

$$\int\limits_G \langle\rho\rangle \cdot \chi_{i_L^G(\rho_L)} = 0$$

for any admissible representation ρ_L of a proper standard Levi subgroup L. By Lemma 4.1 it we have to show that

$$\sum_i (-1)^i \dim Ext^i_G(\rho, i_P^G(\rho_L)) = 0.$$

By Frobenius adjointness we have $Ext^i_G(\rho, i_P^G(\rho_L)) \cong Ext^i_L(r_L^G(\rho),\rho_L)$. The Lemma now follows from the next

Claim 4.3. [B1] *If the center of G is not compact, then for any admissible representations $\rho_1,\rho_2 \in \mathcal{R}(G)$ we have*

$$\sum_i (-1)^i \dim Ext^i_G(\rho_1,\rho_2) = 0.$$

Proof For any $\mathfrak{M} \in Sm(G)$ the space $Ext^i(\mathfrak{M},\rho_2)$ is finite dimensional. Thus the map $[\mathfrak{M}] \mapsto \sum_i (-1)^i \dim Ext^i_G(\mathfrak{M},\rho_2)$ is a well-defined homomorphism $K^0(\mathcal{M}(G)) \to \mathbb{Z}$.

So we will be done if we show that the class of ρ_1, $[\rho_1] \in K^0(\mathcal{M}(G))$ vanishes.

G/G^c is a free abelian group of rank equal to the split rank of the center of G. Hence, if the center of G is not finite, G/G^c is nontrivial, and there exists a nontrivial homomorphism $G \to \mathbb{Z}$ with open kernel. Let us endow $k[\mathbb{Z}]$ with the structure of a smooth G-module by means of this homomorphism. Consider the short exact sequence of G-modules: $0 \to k[\mathbb{Z}] \xrightarrow{[n]\to[n+1]} k[\mathbb{Z}] \to k \to 0$. Tensoring it with ρ_1 we get

$$0 \to \rho_1 \otimes k[\mathbb{Z}] \to \rho_1 \otimes k[\mathbb{Z}] \to \rho_1 \to 0. \quad (47)$$

It is easy to see that $\rho_1 \otimes k[\mathbb{Z}]$ is finitely generated provided ρ_1 is admissible. Thus (47) implies that $[\rho_1] = 0$. $\square$

4.4. *Proof of 0.20.* By Lemma 4.1

$$\sum(-1)^i \dim Ext^i(\rho_1,\rho_2) = \int_G \chi_{\rho_2}\cdot\langle\rho_1\rangle.$$

By Harish-Chandra's Theorem on integrability of characters we can rewrite

$$\int_G \chi_{\rho_2}\cdot\langle\rho_1\rangle = \lim_U \int_U \chi_{\rho_2}\cdot\langle\rho_1\rangle,$$

where U runs over an exhausting family of conjugation-invariant subsets of regular elements. Each U in this family can be written as a disjoint union $U = U^{ell}\cup U^{nonell}$ of a subset of the elliptic (respectively, non-elliptic) set. We have $\int_{U^{nonell}} \chi_{\rho_2}\cdot\langle\rho_1\rangle = 0$, for all U by 4.2, while

$$\int_{U^{ell}} \chi_{\rho_2}\cdot\langle\rho_1\rangle = \int_{U^{ell}/Ad} \chi_{\rho_2}(g)O_g(\langle\rho_1\rangle)d\mu = \int_{U^{ell}/Ad} \chi_{\rho_2}(g)\chi_{\rho_1}(g^{-1})d\mu,$$

where the first equality is (44), and the second follows from 0.19. As U^{ell} increases, the latter expression tends to $\int_{Ell} \chi_{\rho_1}(g^{-1})\chi_{\rho_2}(g)d\mu(g)$. The proof is finished.

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